Mechanically Certifying Formula-based Noetherian Induction Reasoning

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Motivation

Automatisation of induction reasoning:

- large proofs, hard to be checked by humans
- difficulty to certify the underlying code (inference system, orderings,. . .)

☞ (automatized) certification of proof traces by formal certifying tools

Formula-based Noetherian Induction

Noetherian induction principles

Noetherian induction: let $(\mathcal{E}, <)$ be a *well-founded* poset

$$\frac{\forall m \in \mathcal{E}, (\forall k \in \mathcal{E}, k < m \Rightarrow \phi(k)) \Rightarrow \phi(m)}{\forall p \in \mathcal{E}, \phi(p)}$$

☞ $\phi(k)$ are induction hypotheses (IHs)

In a first-order setting, $\mathcal{E}$ can be a set of

- (vector of) terms

$$\frac{\forall \overline{m} \in \mathcal{E}, (\forall \overline{k} \in \mathcal{E}, \overline{k} <_t \overline{m} \Rightarrow \phi(\overline{k})) \Rightarrow \phi(\overline{m})}{\forall \overline{p} \in \mathcal{E}, \phi(\overline{p})}$$

- (first-order) formulas

$$\frac{\forall \gamma \in \mathcal{E}, (\forall \delta \in \mathcal{E}, \delta <_f \gamma \Rightarrow \ ) \Rightarrow}{\forall \rho \in \mathcal{E},}$$

☞ $\phi(\gamma) = \gamma$, $\forall \gamma \in \mathcal{E}$

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Formula-based induction proof techniques

(to recall, $$\frac{\forall \gamma \in \mathcal{E}, (\forall \delta \in \mathcal{E}, \delta <_f \gamma \Rightarrow \delta) \Rightarrow \gamma}{\forall \rho \in \mathcal{E}, \rho}$$)

- inductionless induction ($\mathcal{E}$ has equalities from the proof)
- term-rewriting induction [Reddy, 1990]
- implicit induction [Bronsard *et al.*, 1994], [Bouhoula *et al.*, 1995]

  ☞ generalization of [Reddy, 1990] and of the inductive procedures for conditional equalities from [Kounalis and Rusinowitch, 1990; Bronsard and Reddy, 1991]
- cyclic induction [Stratulat, 2012a]

  ☞ induction performed along *cycles* of formulas

Advantages: lazy induction, mutual induction

Disadvantages: global ordering (at proof or cycle level), cannot be captured by some specific inference rule

Direct relations between term- and formula-based induction principles

Theorem (customizing term- to formula-based proofs)
The term-based induction principle can be represented as a formula-based induction principle.
Proof. If $\mathcal{E}'$ is the set of term vectors for proving $\phi(\overline{x})$, take $\mathcal{E} = \{\phi(\overline{u}) \mid \overline{u} \in \mathcal{E}'\}$ and define $<_f$ as:

$$\phi(\overline{u}) <_f \phi(\overline{v}) \text{ if } \overline{u} <_t \overline{v}$$

Theorem (customizing formula- to term-based proofs)
The formula-based induction principle can be represented as a term-based induction principle when $\mathcal{E}$ is of the form $\{\phi(\overline{t_1}), \ldots, \phi(\overline{t_n})\}$.
Proof. Define $\overline{u} <_t \overline{v}$ if $\phi(\overline{u}) <_f \phi(\overline{v})$.

☞ the general case is conjectured. Translating implicit into explicit induction proofs is *not* satisfactory [Courant, 1996; Kaliszyk, 2005; Nahon *et al.*, 2009]

What about the 'Descente Infinie' ?

☞ contrapositive version of Noetherian induction

(to recall, $\dfrac{\forall m \in \mathcal{E}, (\forall k \in \mathcal{E}, k < m \Rightarrow \phi(k)) \Rightarrow \phi(m)}{\forall p \in \mathcal{E}, \phi(p)}$)

Definition ('Descente Infinie' induction)

$$\frac{\forall m \in \mathcal{E}, \neg\phi(m) \Rightarrow (\exists k \in \mathcal{E}, k < m \wedge \neg\phi(k))}{\forall p \in \mathcal{E}, \phi(p)}$$

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☞ the formula-based version:

$$\frac{\forall \gamma \in \mathcal{E}, \neg\gamma \Rightarrow (\exists \delta \in \mathcal{E}, \delta < \gamma \wedge \neg\delta)}{\forall p \in \mathcal{E}, p}$$

Proof by formula-based induction

$$0 + y = y$$
$$s(u) + v = s(u + v)$$

$\mathcal{E}$:

$$\{z + 0 = z, 0 + 0 = 0, s(x) + 0 = s(x), s(x + 0) = s(x), s(x) = s(x)\}$$

Induction ordering such that

- $s(x + 0) = s(x) <_f s(x) + 0 = s(x)$, $\forall x \in \mathbb{N}$, and
- $x + 0 = x <_f s(x + 0) = s(x)$, $\forall x \in \mathbb{N}$

☞ multiset extension of syntactic orderings (rpo, mpo,. . .)

Proof (à la Descente Infinie).

By contradiction, we assume that $\mathcal{E}$ has a minimal counterexample.

After case analysis, there is no minimal counterexample. □

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Mechanical Proof Certification Methodology

The Coq certification environment

- Coq: proof assistant based on the Calculus of Inductive Constructions (http://coq.inria.fr)
  ☞ integrates Noetherian induction
- proof certification
  ☞ Curry-Howard correspondence:
    - proofs as programs, written in the Gallina language
    - formulas as types

  ☞ proof terms are checked by the kernel

Methodology for certifying formula-based induction reasoning

Idea: explicitly formalize

(1) the induction ordering and the formula weights by means of a syntactic representation of formulas
(2) the formula-based induction principle
(3) the inference steps from the formula-based proof

Advantage: no proof reconstruction techniques are required

Weights for formulas

☞ abstract term algebra: COCCINELLE [Contejean *et al.*, 2007]

- syntactic representation of terms in Coq

  Inductive **term** : Set :=
  | Var : variable → **term**
  | Term : symbol → list **term** → **term**

Defining induction orderings in COCCINELLE

```
Inductive rpo (bb : nat) : term → term → Prop :=
    | Subterm : ∀ f l t s, mem equiv s l → rpo_eq bb t s → rpo bb t (Term f l)
    | Top_gt :
       ∀ f g l l', prec P g f → (∀ s', mem equiv s' l' → rpo bb s' (Term f l)) →
          rpo bb (Term g l') (Term f l)
    | Top_eq_lex :
       ∀ f g l l', status P f = Lex → status P g = Lex → prec_eq P f g → (length
l = length l' ∨ (length l' ≤ bb ∧ length l ≤ bb)) → rpo_lex bb l' l →
          (∀ s', mem equiv s' l' → rpo bb s' (Term g l)) →
          rpo bb (Term f l') (Term g l)
    | Top_eq_mul :
       ∀ f g l l', status P f = Mul → status P g = Mul → prec_eq P f g →
rpo_mul bb l' l →
          rpo bb (Term f l') (Term g l)

    with rpo_mul ( bb : nat) : list term → list term → Prop :=
    | List_mul : ∀ a lg ls lc l l',
       permut0 equiv l' (ls ++ lc) → permut0 equiv l (a :: lg ++ lc) →
       (∀ b, mem equiv b ls → ∃ a', mem equiv a' (a :: lg) ∧ rpo bb b a') →
       rpo_mul bb l' l.
```

Notation less := (**rpo_mul** (bb)).

Defining Coq specification and translation functions

```
Fixpoint plus (x y:nat): nat :=
match x with
| O ⇒ y
| (S x') ⇒ S (plus x' y)
end.
```

- COCCINELLE symbols: id_0, id_S, id_plus
  ☞ precedence and status
- translation function for any natural into a COCCINELLE term

```
Fixpoint model_nat (v: nat): term :=
match v with
| O ⇒ (Term id_0 nil)
| (S x) ⇒ let r := model_nat x in (Term id_S (r::nil))
end.
```

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Defining the set $\mathcal{E}$ and formula weights from a Spike proof

- syntactically represent each conjecture ϕ as a weight w_ϕ
- the variables are shared using anonymous functions

$$\texttt{fun}\ \overline{x} \Rightarrow (\phi, w_\phi)$$

- $\mathcal{E}'$ will consist of anonymous functions

Example

$\mathcal{E}'$: {(fun *u1* ⇒ ((plus *u1* 0) = *u1* , w_1::w_2::nil),...}, where

- w_1 is (Term id_plus ((model_nat *u1*):: (Term id_0 nil):: nil))
- w_2 is model_nat *u1*

- $\mathcal{E}$ is computed from $\mathcal{E}'$

Formalizing the formula-based induction principle

☞ COCCINELLE extended with dual computable function for 'less '

Adding lemmas showing

- its equivalence with 'less '
- properties (well-foundedness, stability)

Specifying the formula-based induction principle

(to recall, $$\frac{\forall \gamma \in \mathcal{E}, (\forall \delta \in \mathcal{E}, \delta <_f \gamma \Rightarrow \delta) \Rightarrow \gamma}{\forall \rho \in \mathcal{E}, \rho}$$)

(1) (main lemma)
∀ *F*, In *F* $\mathcal{E}'$ → ∀ *u1*, (∀ *F'*, In *F'* $\mathcal{E}'$ → ∀ *e1*, less (snd (*F' e1*)) (snd (*F u1*)) → fst (*F' e1*)) → fst (*F u1*).

(2) (all_true lemma)
∀ *F*, In *F* $\mathcal{E}'$ → ∀ *u1*: **nat**, fst (*F u1*).

☞ (2) is derived from (1) using Coq's Noetherian induction

Proving the main lemma

☞ the anonymous functions from $\mathcal{E}'$ are treated independently, one-by-one.

the conjecture of each anonymous function may be proved using (instances of) other conjectures that are

- logically equivalent (deductive reasoning)
- smaller

Proving logical equivalences

- variable instantiations are controlled by Coq functional schemas [Barthe and Courtieu, 2002]

Example (x is replaced by 0 and $(S\ z)$ using f)

```
Fixpoint f (x: nat) {struct x} : nat :=
 match x with
| 0 ⇒ 0
| (S z) ⇒ 0
end.

Functional Scheme f_ind := Induction for f Sort Prop.
```

The instances are generated by the Coq script

```
pattern x, (f x).  apply f_ind.
```

One-to-one translations

- Equality reasoning using rewriting
  - rewriting $C[f(t)]$ with $f(x) = \ldots$ yields
    `pattern` t`.  simpl` f`. cbv  beta.`
    - `pattern` t isolates t from C,
    - `simpl` f rewrites $f(t)$,
    - `cbv beta` puts back the resulted term in C.
- Tautologies (of the form $t = t$) are proved using `auto.`

Weight comparisons

User-defined tacticals for automatization:

- rewrite with model functions
- compute the ordering

(1) terms of the form $(model\_sort\ (f\ x_1 \cdots x_n))$ will be replaced by $(Id\_f\ (model\_sort\ x_1) \cdots (model\_sort\ x_n))$

(2) the replacement of terms of the form (model_*sort* t) with COCCINELLE abstraction variables of the form (*Var* i), $i \in \mathbb{N}$;

(3) computing by reflection the comparison result of weights with abstraction variables;

(4) the use of stability property of 'less ' to compare with abstraction variables instead of original weights.

Examples

Implicit induction inference systems

- inference rules: transitions between states

$$(conjectures, premises)$$

☞ premises are 'previous' conjectures with no minimal counterexamples (w.r.t. $<_f$).

- derivation of E^0 with an inference system I:
$(E^0,\ \emptyset) \vdash_I (E^1,\ H^1) \vdash_I \ldots$

- proof: finite derivation whose last state has no conjectures:
$(E^0,\ \emptyset) \vdash_I (E^1,\ H^1) \vdash_I \ldots \vdash_I (\emptyset, H^n)$

The concrete inference system I_{imp}

☞ Ax are axioms oriented into rewrite rules

GenNat (G): $(E \cup \{\phi\langle x \rangle\},\ H) \vdash_{I_{imp}} (E \cup \{\phi_1, \phi_2\},\ H \cup \{\phi\})$,
where $\phi\{x \mapsto 0\} \to_{Ax} \phi_1$, $\phi\{x \mapsto s(x')\} \to_{Ax} \phi_2$

SimpEq (S): $(E \cup \{\phi\},\ H) \vdash_{I_{imp}} (E \cup \{\psi\},\ H)$,
if $\phi \to_{Ax \cup (E \cup H)_{\leq_f \phi}} \psi$

ElimTaut (E): $(E \cup \{\phi\}, H) \vdash_{I_{imp}} (E,\ H)$,
if ϕ is a tautology

An I_{imp}-proof of $x + 0 = x$

Rewrite rules

$$0 + y \to y$$

$$s(u) + v \to s(u + v)$$

I_{imp}-proof of $x + 0 = x$:

$(\{x + 0 = x\}, \emptyset)$

$\vdash^{G}_{I_{imp}} (\{0 = 0, s(x' + 0) = s(x')\}, \{x + 0 = x\})$

$\vdash^{S}_{I_{imp}} (\{0 = 0, s(x') = s(x')\}, \{x + 0 = x\})$

$\vdash^{E(2)}_{I_{imp}} (\emptyset, \{x + 0 = x\})$

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ElimTaut (*E*): $(E \cup \{\phi\}, H) \vdash_{I_{imp}} (E,\ H)$,
if ϕ is a tautology

Certifying the I_{imp}-proof of $x + 0 = x$

- ordering

```
Definition index (f:symb) :=
  match f with
  | id_0 ⇒ 2
  | id_S ⇒ 3
  | id_plus ⇒ 7
  end.
```

```
Definition status (f:symb) :=
  match f with
  | id_0 ⇒ rpo.Mul
  | id_S ⇒ rpo.Mul
  | id_plus ⇒ rpo.Mul

  end.
```

- list of anonymous functions

```
Definition type_LF := nat → Prop × List.list term.
Definition ℰ' := [F_1, F_2, F_3, F_4].
(* for all equalities from the proof *)
```

```
Definition F_1 : type_LF:= (fun u1 ⇒ ((plus u1 0) = u1 ,
(Term id_plus ((model_nat u1):: (Term id_0
nil)::nil))::(model_nat u1)::nil)).

Definition F_2 : type_LF:= (fun _ ⇒ (0 = 0, (Term id_0
nil)::(Term id_0 nil)::nil)).

Definition F_3 : type_LF:= (fun u2 ⇒ ((S (plus u2 0)) = (S
u2), (Term id_S ((Term id_plus ((model_nat u2):: (Term id_0
nil)::nil))::nil))::(Term id_S ((model_nat u2)::nil))::nil)).

Definition F_4 : type_LF:= (fun u2 ⇒ ((S u2) = (S u2),
(Term id_S ((model_nat u2)::nil) )::(Term id_S ((model_nat
u2)::nil))::nil)).
```

Proof of the main lemma

$\forall$ *F*, In *F* $\mathcal{E}'$ $\rightarrow$ $\forall$ *u1*, ($\forall$ *F'*, In *F'* $\mathcal{E}'$ $\rightarrow$ $\forall$ *e1*, less (snd (*F' e1*)) (snd (*F u1*)) $\rightarrow$ fst (*F' e1*)) $\rightarrow$ fst (*F u1*).

Proof.

By case analysis.

- F_1 (recall, (plus *u1* 0) = *u1*): instantiate *u1* by
  `pattern u1, (f u1).`
  - case *u1* is 0: by `auto.`
  - case *u1* is S *u2*: choose as IH
    F_3 (recall, S (plus *u2* 0) = (S *u2*)), then simplify
- F_2 (recall, 0=0): by `auto.`
- F_3: choose as IH F_1, then simplify
- F_4 (recall, (S *u2*) = (S *u2*)): by `auto.`

□

Discussions

Implicit induction reasoning:

- easily automatized (Spike, RRL)
- generate large Spike proofs
  - validation of the JavaCard platform [Barthe and Stratulat, 2003]

instruction	proved	lemmas	Generate	U. R.	C. R.	Subsumption	Taut.	time
ACONST_NULL	yes	0	0	4	1	0	1	0.5s
ALOAD	n.y.	0	0	0	0	0	0	0.0
ARITH	yes	33	100	8771	2893	979	2178	8m

  - validation of telecommunication protocols[Rusinowitch *et al.*, 2003] ☞ 40% of the lemmas are automatically certified

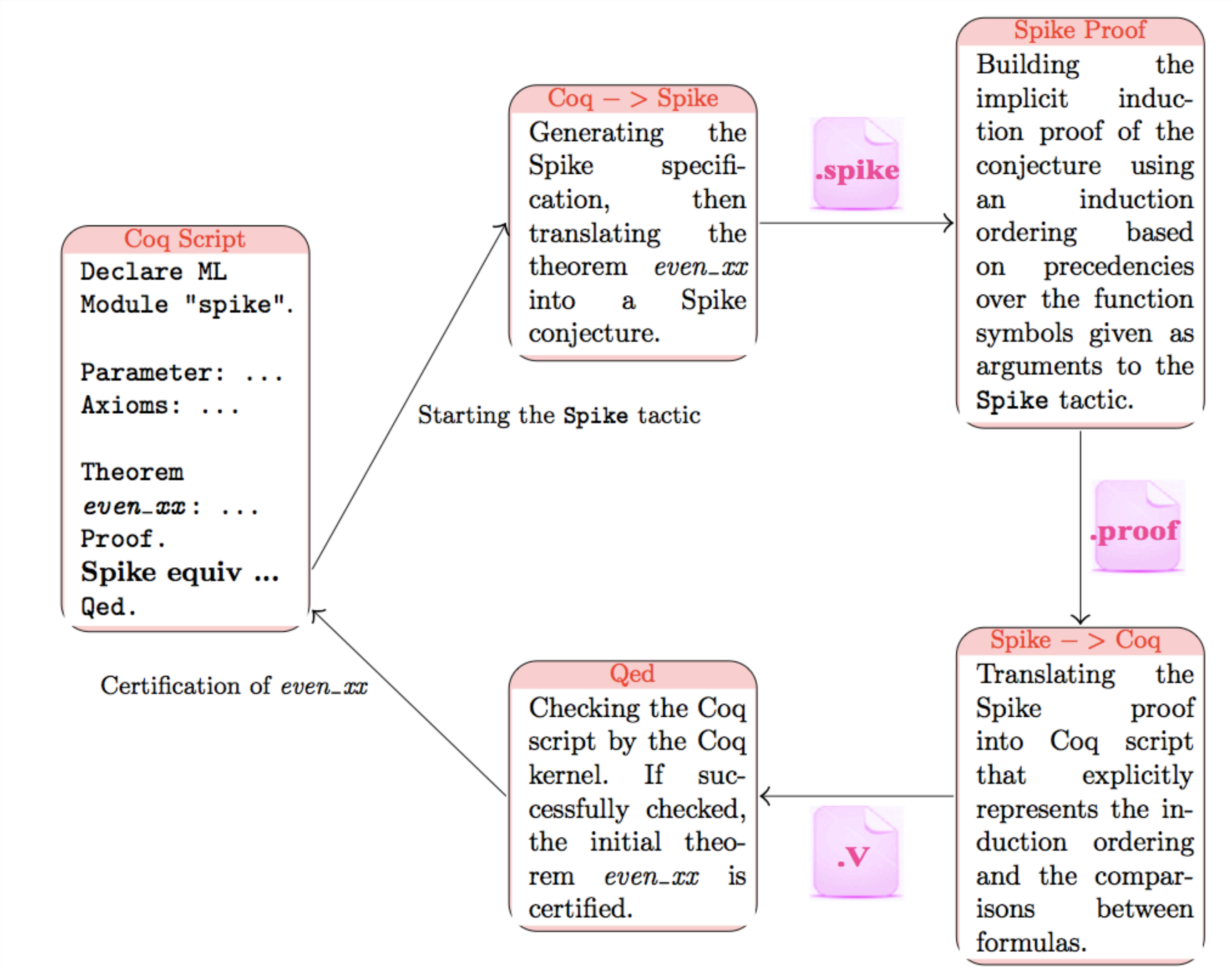
The certification process may be less effective

- check every reductive ordering constraint
  ☞ multiple calls to COCCINELLE functions
- check every formula from the proof
  ☞ large $\mathcal{E}'$ sets.

The Coq tactic Spike

☞ solves the translation problems at specification level

```
Theorem even_xx: ∀ x, even (add (x  x)) = true.
 Proof.
 Spike    equiv [[even, odd]]
          greater [ [even, true ,false, S , 0, add],
                             [ add, S, 0] ].
 Qed.
```



Cyclic reasoning on one slide

☞ non-reductive reasoning

$$0 + y = y$$
$$s(u) + v = s(u + v)$$

$\mathcal{E}$: $\{z + 0 = z, 0 + 0 = 0, s(x) + 0 = s(x)\}$

☞ less elements in $\mathcal{E}$

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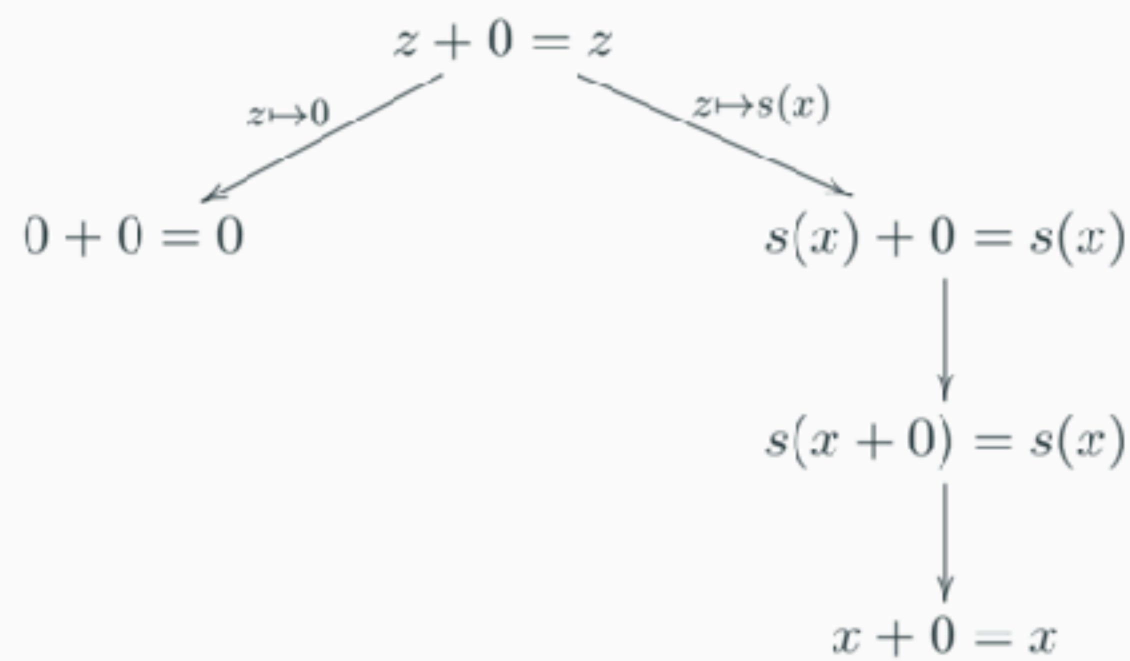

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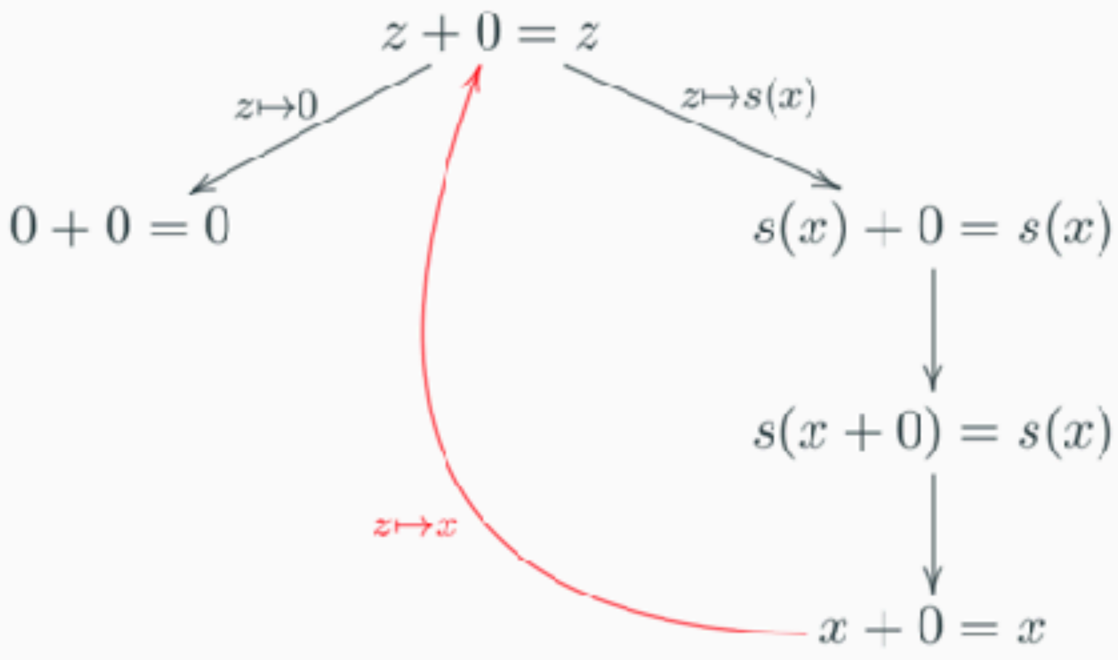

$$x + 0 = x <_f s(x) + 0 = s(x)$$

$\mathcal{E}$: $\{z + 0 = z, 0 + 0 = 0, s(x) + 0 = s(x)\}$

☞ less elements in $\mathcal{E}$

Conclusions and Future Work

Conclusions

- methodology for automatically certifying any formula-based induction proof
  ☞ implicit induction, cyclic induction

- automatic Coq certification of Spike's implicit induction proofs
  ☞ Coq checkpoints for Spike specifications and proofs:
  (1) (ground) convergence and completeness properties: acceptance of the translated functions by Coq
  (2) variable instantiation schemas: functional schemes
  (3) certifying the induction principle: the main lemma

  ☞ limited Spike specifications + control in the automatic translation of the proofs

Future Work

- Spike proof certification : allow more general specifications and inference rules
  ☞ certifying reductive-free cyclic proofs
- building a formula-based induction proof environment directly in Coq
  - for lazy reasoning and cyclic induction
  - for automatically performing implicit induction
    ☞ direct use of Coq tactics and no translation
- dissemination and implementation for other proof environments (Isabelle/HOL, PVS, . . .)

More information at

- recent article (2017)

  `S. Stratulat. Mechanically certifying formula-based Noetherian induction reasoning. Journal of Symbolic Computation, 41 pages.`

- http://code.google.com/p/spike-prover/

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Thank you !

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