Finite and infinite traces, inductively and coinductively

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Overview

- Classic fact: if an LTS is image-finite, then finite trace equivalence coincides with infinite trace equivalence
- 'Standard' proof: inductively construct infinite paths
- This talk: coinductive proof – basic exercise in coinduction
- Idea from (Bonsangue/Rot/Ancona/de Boer/Rutten, ICALP 2014), where it is a little bit hidden
- Related to König's lemma, which was done coinductively in Isabelle (Lochbihler and/or Hölzl and/or ...?)

Warming up: König's tree lemma

Lemma
Suppose t is a finitely branching tree whose root has infinitely many successors. Then t has an infinite path.


König's Tree Lemma

*This article was **Featured Proof** between 2nd May 2011 and 5th May 2012.*

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Standard approach: explicitly construct an infinite path, see e.g. the three proofs at https://proofwiki.org/wiki/König%27s_Tree_Lemma

Coinduction in a lattice

$b \colon L \to L$ monotone function on complete lattice L:

$$\frac{y \leq x \leq b(x)}{y \leq \nu b} \text{ coinduction}$$

Trees with infinite paths

Let

$$T = \{t \mid t \text{ is (the root of) a finitely branching tree}\}$$

and $\mathcal{P}(T)$ the powerset; complete lattice, ordered by inclusion.

Define $p\colon \mathcal{P}(T) \to \mathcal{P}(T)$ by

$$p(S) = \{t \mid \exists t'.t \to t' \text{ and } t' \in S\}$$

Then

$$\nu p = \{t \in T \mid t \text{ has an infinite path}\}$$

(this is where the explicit construction of paths comes in).

König's tree lemma revisited

Let

$$I = \{t \in T \mid t \text{ has infinitely many successors}\}$$

König's lemma reformulated:

$$I \subseteq \nu p$$

To prove this, it suffices to show

$$I \subseteq p(I)$$

This is the essence: if t has infinitely many successors and finite branching, then one of it's children has infinitely many successors.

Separation of concerns:

- characterisation νp ("inductive" construction of infinite paths)
- essence of the proof (selection of successor) is coinductive

LTSs, traces

Labelled transition system (LTS): set X with relation $\rightarrow \subseteq X \times A \times X$

Finitely branching if for all x: the set $\{x' \mid \exists a. x \xrightarrow{a} x'\}$ is finite
Image-finite if for all x, a the set $\{x' \mid x \xrightarrow{a} x'\}$ is finite

Finite words/traces denoted by A^*, infinite words/traces by A^ω

Statement

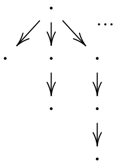
Denote by $\mathrm{tr}_{\mathrm{fin}}(x) \subseteq A^*$ the set of traces starting in x, and $\mathrm{tr}_{\mathrm{inf}}(x) \subseteq A^\omega$ the set of infinite traces.

Theorem

Suppose our LTS is image-finite. Then for any $x \in X$: if $\mathrm{tr}_{\mathrm{fin}}(x) \subseteq \mathrm{tr}_{\mathrm{fin}}(y)$, then $\mathrm{tr}_{\mathrm{inf}}(x) \subseteq \mathrm{tr}_{\mathrm{inf}}(y)$

"Standard" proof: explicitly construct traces by induction

Image-finiteness needed:



Trace semantics, more precisely

Note that for any X, Y, the set $\mathcal{P}(Y)^X$ is a complete lattice, ordered by pointwise inclusion.

Finite trace semantics: least map $\mathrm{tr}_{\mathrm{fin}} \colon X \to \mathcal{P}(A^*)$ such that

- $\varepsilon \in \mathrm{tr}_{\mathrm{fin}}(x)$ for all x
- if $x \xrightarrow{a} x'$ and $w \in \mathrm{tr}_{\mathrm{fin}}(x')$ then $aw \in \mathrm{tr}_{\mathrm{fin}}(x)$

Infinite trace semantics: greatest map $\mathrm{tr}_{\mathrm{inf}} \colon X \to \mathcal{P}(A^\omega)$ such that for all $x \in X$, $a \in A$, $w \in A^\omega$:

- if $a\sigma \in \mathrm{tr}_{\mathrm{inf}}(x)$ then $\exists x'.\ x \xrightarrow{a} x'$ and $\sigma \in \mathrm{tr}_{\mathrm{inf}}(x')$.

Infinite trace semantics is coinductive, but trace equivalence not (I think), so need a trick to prove the theorem

Infinite traces from finite traces

Define $\mathrm{pref}\colon A^\omega \to \mathcal{P}(A^*)$

$$\mathrm{pref}(\sigma) = \{w \mid w \prec \sigma\}$$

where $\prec$ is the prefix relation. (This is finite trace semantics of a canonical LTS on A^ω.)

Let $\mathrm{pref}^{-1}\colon \mathcal{P}(A^*) \to \mathcal{P}(A^\omega)$ be given by

$$\mathrm{pref}^{-1}(S) = \{\sigma \mid w \in S \text{ for all } w \text{ with } w \prec \sigma\}\,.$$

We will prove:

Theorem

On image-finite LTSs: $\mathrm{tr}_{\mathrm{inf}} = \mathrm{pref}^{-1} \circ \mathrm{tr}_{\mathrm{fin}}\,.$

Proof

Theorem
On image-finite LTSs: $\mathrm{tr}_{\mathrm{inf}} = \mathrm{pref}^{-1} \circ \mathrm{tr}_{\mathrm{fin}}$.

Start with $\mathrm{tr}_{\mathrm{inf}} \subseteq \mathrm{pref}^{-1} \circ \mathrm{tr}_{\mathrm{fin}}$.

"If x accepts an infinite trace σ, then also all its finite prefixes"

Bit more precisely: prove that $\forall n \in \mathbb{N}, \sigma \in A^\omega, x \in X$:

$$\sigma \in \mathrm{tr}_{\mathrm{inf}}(x) \to \sigma|_n \in \mathrm{tr}_{\mathrm{fin}}(x)$$

by induction on n, where $\sigma|_n$ is the prefix of σ of length n.

Proof (2)

Theorem

On image-finite LTSs: $\mathrm{tr}_{\mathrm{inf}} = \mathrm{pref}^{-1} \circ \mathrm{tr}_{\mathrm{fin}}$.

Now, we prove $\mathrm{tr}_{\mathrm{inf}} \supseteq \mathrm{pref}^{-1} \circ \mathrm{tr}_{\mathrm{fin}}$: the interesting bit.

We can use that $\mathrm{tr}_{\mathrm{inf}}$ is defined coinductively!

Suffices to prove that for all $x \in X$, $a \in A$, $\sigma \in A^\omega$:

- if $a\sigma \in \mathrm{pref}^{-1} \circ \mathrm{tr}_{\mathrm{fin}}(x)$ then $\exists x'.\ x \xrightarrow{a} x'$ and $\sigma \in \mathrm{pref}^{-1} \circ \mathrm{tr}_{\mathrm{fin}}(x')$.

To see this:

- If $a\sigma \in \mathrm{pref}^{-1} \circ \mathrm{tr}_{\mathrm{fin}}(x)$, then all finite prefixes of $a\sigma$ are in $\mathrm{tr}_{\mathrm{fin}}(x)$
- Since there are finitely many a-successors (x' such that $x \xrightarrow{a} x'$) there is one s.t. $w \in \mathrm{tr}_{\mathrm{fin}}(x')$ for infinitely many prefixes w of σ
- Since $\mathrm{tr}_{\mathrm{fin}}(x')$ is prefix-closed, it follows that *all* prefixes of σ are in $\mathrm{tr}_{\mathrm{fin}}(x')$
- Hence $\sigma \in \mathrm{pref}^{-1} \circ \mathrm{tr}_{\mathrm{fin}}(x')$.

Finite and infinite traces

We established:

Theorem

On image-finite LTSs: $\mathrm{tr}_{\mathrm{inf}} = \mathrm{pref}^{-1} \circ \mathrm{tr}_{\mathrm{fin}}$.

hence it easily follows that $\mathrm{tr}_{\mathrm{fin}}(x) \subseteq \mathrm{tr}_{\mathrm{fin}}(y) \to \mathrm{tr}_{\mathrm{inf}}(x) \subseteq \mathrm{tr}_{\mathrm{inf}}(y)$ as desired.

Once again (like in König's case) there is a separation of concerns:

- coinductive characterisation of infinite trace acceptance (no explicit paths)
- coinductive proof of the main point (selection of successors)

Alternative: final sequence argument

Infinite trace semantics $\mathrm{tr}_{\mathrm{inf}}$ is defined as the greatest fixed point of a map $\varphi \colon \mathcal{P}(A^\omega)^X \to \mathcal{P}(A^\omega)^X$, which one may compute using the (ordinal-indexed) final sequence:

$$\top \geq \varphi(\top) \geq \varphi(\varphi(\top)) \geq \dots$$

- States $x, y \in X$ are finite trace equivalent if $\varphi^i(\top)(x) = \varphi^i(\top)(y)$ for every $i < \omega$
- If φ is cocontinuous then $\nu\varphi = \bigwedge_{i<\omega} \varphi^i(\top)$

Similar classical argument for bisimilarity (on image-finite systems) and its approximants

Coalgebraic picture

Image-finite LTS is a coalgebra of the form

$$f \colon X \to (\mathcal{P}_f X)^A$$

Finitely branching LTS is a coalgebra of the form

$$f \colon X \to \mathcal{P}_f(A \times X)$$

- Since $\mathcal{P}_f(A \times -)$ is finitary, it follows from (Hasuo/Cho/Kataoka/Jacobs, MFPS 2013) that the final sequence of φ (computing the infinite traces) stabilises at ω.
- For image-finite LTS, this doesn't seem to work (?)
- Systematic coalgebraic picture of finite vs. infinite trace semantics still lacking

In our ICALP 2014 paper: original coinductive proof presented a bit more generally; works at least for tree automata.

Conclusion

- Coinductive proof that finite trace equivalence implies infinite trace equivalence (König's lemma-type arguments)
- Separates coinductive characterisation (and its ‘correctness’) from actual argument