

## Superposition with Lambdas

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**Abstract** We designed a superposition calculus for a clausal fragment of extensional polymorphic higher-order logic that includes anonymous functions but excludes Booleans. The inference rules work on  $\beta\eta$ -equivalence classes of  $\lambda$ -terms and rely on higher-order unification to achieve refutational completeness. We implemented the calculus in the Zipperposition prover and evaluated it on TPTP and Isabelle benchmarks. The results suggest that superposition is a suitable basis for higher-order reasoning.

**Keywords** superposition calculus · higher-order logic · refutational completeness

### 1 Introduction

Superposition [6] is widely regarded as the calculus par excellence for reasoning about first-order logic with equality. To increase automation in proof assistants and other verification tools based on higher-order formalisms, we propose to generalize superposition to an extensional, polymorphic, clausal version of higher-order logic (also called simple type theory). Our ambition is to achieve a *graceful* extension, which coincides with standard superposition on first-order problems and smoothly scales up to arbitrary higher-order problems.

Bentkamp, Blanchette, Cruanes, and Waldmann [13] designed a family of superposition-like calculi for a  $\lambda$ -free clausal fragment of higher-order logic, with currying and applied variables. We adapt their extensional nonpurifying calculus to support  $\lambda$ -terms (Sect. 3). Our calculus does not support interpreted Booleans; it is conceived as the penultimate milestone towards a superposition calculus for full higher-order logic. If desired, Booleans can be encoded in our logic fragment using an uninterpreted type and uninterpreted “proxy” symbols corresponding to equality, the connectives, and the quantifiers.

Designing a higher-order superposition calculus poses three main challenges:

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1. Standard superposition is parameterized by a ground-total simplification order  $\succ$ , but such orders do not exist for  $\lambda$ -terms considered equal up to  $\beta$ -conversion. The relations designed for proving termination of higher-order term rewriting systems, such as HORPO [42] and CPO [24], lack many of the desired properties (e.g., transitivity, stability under grounding substitutions).
2. Higher-order unification is undecidable and may give rise to an infinite set of incomparable unifiers. For example, the constraint  $f(y a) \stackrel{?}{=} y(f a)$  admits infinitely many independent solutions of the form  $\{y \mapsto \lambda x. f^n x\}$ .
3. In first-order logic, to rewrite into a term  $s$  using an oriented equation  $t \approx t'$ , it suffices to find a subterm of  $s$  that is unifiable with  $t$ . In higher-order logic, this is insufficient. Consider superposition from  $f c \approx a$  into  $y c \approx y b$ . The left-hand sides can obviously be unified by  $\{y \mapsto f\}$ , but the more general  $\{y \mapsto \lambda x. z x(f x)\}$  also gives rise to a subterm  $f c$  after  $\beta$ -reduction. The corresponding inference generates the clause  $z c a \approx z b(f b)$ .

To address the first challenge, we adopt the  $\eta$ -short  $\beta$ -normal form to represent  $\beta\eta$ -equivalence classes of  $\lambda$ -terms. In the spirit of Jouannaud and Rubio's early joint work [41], we state requirements on the term order only for ground terms (i.e., closed monomorphic  $\beta\eta$ -equivalence classes); the nonground case is connected to the ground case via stability under grounding substitutions. Even on ground terms, we cannot obtain all desirable properties. We sacrifice compatibility with arguments (the property that  $s' \succ s$  implies  $s' t \succ s t$ ), compensating with an *argument congruence* rule (ARGCONG), as in Bentkamp et al. [13].

For the second challenge, we accept that there might be infinitely many incomparable unifiers and enumerate a complete set (including the notorious flex–flex pairs [39]), relying on heuristics to keep the combinatorial explosion under control. The saturation loop must also be adapted to interleave this enumeration with the theorem prover's other activities (Sect. 6). Despite its reputation for explosiveness, higher-order unification is a conceptual improvement over SK combinators, because it can often *compute* the right unifier. Consider the conjecture  $\exists z. \forall x y. z x y \approx f y x$ . After negation, classification, and skolemization (which are as for first-order logic), the formula becomes  $z(\text{sk}_x z)(\text{sk}_y z) \approx f(\text{sk}_y z)(\text{sk}_x z)$ . Higher-order unification quickly computes the unique unifier:  $\{z \mapsto \lambda x y. f y x\}$ . In contrast, an encoding approach based on combinators, similar to the one implemented in Sledgehammer [52], would blindly enumerate all possible SK terms for  $z$  until the right one,  $S(K(S f))K$ , is found. Given the definitions  $S z y x \approx z x(y x)$  and  $K x y \approx x$ , the E prover [58] in *auto* mode needs to perform 3757 inferences to derive the empty clause.

For the third challenge, the idea is that, when applying  $t \approx t'$  to perform rewriting inside a higher-order term  $s$ , we can encode an arbitrary context as a fresh higher-order variable  $z$ , unifying  $s$  with  $z t$ ; the result is  $(z t')\sigma$ , for some unifier  $\sigma$ . This is performed by a dedicated *fluid subterm superposition* rule (FLUIDSUP).

Functional extensionality is also considered a challenge for higher-order reasoning [15], although similar difficulties arise with first-order sets and arrays [36]. Our approach is to add extensionality as an axiom and provide optional rules as optimizations (Sect. 5). With this axiom, our calculus is refutationally complete w.r.t. extensional Henkin semantics (Sect. 4). Our proof employs the new saturation framework by Waldmann et al. [67] to derive dynamic completeness of a given clause prover from ground static completeness.

We implemented the calculus in the Zipperposition prover [30] (Sect. 6). Our empirical evaluation includes benchmarks from the TPTP [62] and interactive verification problems exported from Isabelle/HOL [25] (Sect. 7). The results appear promising and suggest that an implementation inside a high-performance prover such as E [58] or Vampire [48] could compete against the rapidly evolving state of the art (Sect. 8).

An earlier version of this article was presented at the 2019 edition of CADE [11]. This article extends the conference paper with more explanations, detailed soundness and completeness proofs, including dynamic completeness, and new optional inference rules. We have also updated the empirical evaluation and extended the coverage of related work. Finally, we tightened side condition 4 of FLUIDSUP, making the rule slightly less explosive.

## 2 Logic

Our extensional polymorphic clausal higher-order logic is a restriction of full TPTP THF [17] to rank-1 (top-level) polymorphism, as in TH1 [43]. In keeping with standard superposition, we consider only formulas in conjunctive normal form, without explicit quantifiers or Boolean type. We use Henkin semantics [16,33,37], as opposed to the standard semantics that serves as the foundation of the HOL systems [35]. By admitting nonstandard models, Henkin semantics is not subject to Gödel's first incompleteness theorem, allowing us to claim not only soundness but also refutational completeness of our calculus.

**Syntax** We fix a set  $\Sigma_{\text{ty}}$  of type constructors with arities and a set  $\mathcal{V}_{\text{ty}}$  of type variables. We require at least one nullary type constructor and a binary function type constructor  $\rightarrow$  to be present in  $\Sigma_{\text{ty}}$ . A type  $\tau, \nu$  is either a type variable  $\alpha \in \mathcal{V}_{\text{ty}}$  or has the form  $\kappa(\bar{\tau}_n)$  for an  $n$ -ary type constructor  $\kappa \in \Sigma_{\text{ty}}$  and types  $\bar{\tau}_n$ . We use the notation  $\bar{a}_n$  or  $\bar{a}$  to stand for the tuple  $(a_1, \dots, a_n)$  or product  $a_1 \times \dots \times a_n$ , where  $n \geq 0$ . We write  $\kappa$  for  $\kappa()$  and  $\tau \rightarrow \nu$  for  $\rightarrow(\tau, \nu)$ . Type declarations have the form  $\Pi \bar{a}_m. \tau$  (or simply  $\tau$  if  $m = 0$ ), where all type variables occurring in  $\tau$  belong to  $\bar{a}_m$ .

We fix a set  $\Sigma$  of (function) symbols  $a, b, c, f, g, h, \dots$ , with type declarations, written as  $f : \Pi \bar{a}_m. \tau$  or  $f$ , and a set  $\mathcal{V}$  of term variables with associated types, written as  $x : \tau$  or  $x$ . The notation  $t : \tau$  will also be used to indicate the type of arbitrary terms  $t$ . We require the presence of a symbol of type  $\Pi \alpha. \alpha$  and a symbol  $\text{diff} : \Pi \alpha, \beta. (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha$  in  $\Sigma$ . We use  $\text{diff}$  to express the polymorphic functional extensionality axiom.

A signature is a tuple  $(\Sigma_{\text{ty}}, \mathcal{V}_{\text{ty}}, \Sigma, \mathcal{V})$ . The set of *raw  $\lambda$ -terms* is defined inductively as follows. Every  $x : \tau \in \mathcal{V}$  is a raw  $\lambda$ -term of type  $\tau$ . If  $f : \Pi \bar{a}_m. \tau \in \Sigma$  and  $\bar{v}_m$  is a tuple of types, called *type arguments*, then  $f\langle \bar{v}_m \rangle$  (or simply  $f$  if  $m = 0$ ) is a raw  $\lambda$ -term of type  $\tau\{\bar{a}_m \mapsto \bar{v}_m\}$ . If  $x : \tau$  and  $t : \nu$ , then the  *$\lambda$ -expression*  $\lambda x. t$  is a raw  $\lambda$ -term of type  $\tau \rightarrow \nu$ . If  $s : \tau \rightarrow \nu$  and  $t : \tau$ , then the *application*  $st$  is a raw  $\lambda$ -term of type  $\nu$ .

A raw  $\lambda$ -term  $s$  is a *subterm* of a raw  $\lambda$ -term  $t$ , written  $t = t[s]$ , if  $t = s$ , if  $t = (\lambda x. u[s])$ , if  $t = (u[s])v$ , or if  $t = u(v[s])$  for some raw  $\lambda$ -terms  $u$  and  $v$ . A *proper* subterm of a raw  $\lambda$ -term  $t$  is any subterm of  $t$  that is distinct from  $t$  itself.

The  $\alpha$ -renaming rule is defined as  $(\lambda x. t) \rightarrow_{\alpha} (\lambda y. t\{x \mapsto y\})$ , where  $y$  does not occur free in  $t$  and is not captured by a  $\lambda$ -binder in  $t$ . Raw  $\lambda$ -terms form equivalence classes modulo  $\alpha$ -renaming, called  *$\lambda$ -terms*. A variable occurrence is *free* in a  $\lambda$ -term if it is not bound by a  $\lambda$ -expression. A  $\lambda$ -term is *ground* if it is built without using type variables and contains no free term variables. Using the spine notation [28],  $\lambda$ -terms can be decomposed in a unique way as a non-application *head*  $t$  applied to zero or more arguments:  $t s_1 \dots s_n$  or  $t \bar{s}_n$  (abusing notation).

The  $\beta$ - and  $\eta$ -reduction rules are specified on  $\lambda$ -terms as  $(\lambda x. t) u \rightarrow_{\beta} t\{x \mapsto u\}$  and  $(\lambda x. t x) \rightarrow_{\eta} t$ . For  $\beta$ , bound variables in  $t$  are implicitly renamed to avoid capture; for  $\eta$ , the variable  $x$  may not occur free in  $t$ . The  $\lambda$ -terms form equivalence classes modulo  $\beta\eta$ -reduction, called  *$\beta\eta$ -equivalence classes* or simply *terms*. When defining operations that need to analyze the structure of terms, we will use the  $\eta$ -short  $\beta$ -normal form  $t \downarrow_{\beta\eta}$ , obtained

by applying  $\rightarrow_\beta$  and  $\rightarrow_\eta$  exhaustively, as a representative of the equivalence class  $t$ . In particular, we lift the notions of subterms and occurrences of variables to  $\beta\eta$ -equivalence classes via their  $\eta$ -short  $\beta$ -normal representative. Many authors prefer the  $\eta$ -long  $\beta$ -normal form [39, 41, 51], but in a polymorphic setting it has the drawback that instantiating a type variable with a functional type can lead to  $\eta$ -expansion. We reserve the letters  $s, t, u, v$  for terms and  $x, y, z$  for variables.

An equation  $s \approx t$  is formally an unordered pair of terms  $s$  and  $t$ . A literal is an equation or a negated equation, written  $\neg s \approx t$  or  $s \not\approx t$ . A clause  $L_1 \vee \dots \vee L_n$  is a finite multiset of literals  $L_j$ . The empty clause is written as  $\perp$ .

In general, a substitution  $\{\bar{\alpha}_m, \bar{x}_n \mapsto \bar{v}_m, \bar{s}_n\}$ , where each  $x_j$  has type  $\tau_j$  and each  $s_j$  has type  $\tau_j\{\bar{\alpha}_m \mapsto \bar{v}_m\}$ , maps  $m$  type variables to  $m$  types and  $n$  term variables to  $n$  terms. The letters  $\theta, \pi, \rho, \sigma$  are reserved for substitutions. Substitutions  $\alpha$ -rename terms to avoid capture; for example,  $(\lambda x. y)\{y \mapsto x\} = (\lambda x'. x)$ . The composition  $\rho\sigma$  applies  $\rho$  first:  $t\rho\sigma = (t\rho)\sigma$ . The notation  $\sigma[\bar{x}_n \mapsto \bar{s}_n]$  denotes the substitution that replaces each  $x_i$  by  $s_i$  and that otherwise coincides with  $\sigma$ . A *complete set of unifiers* on a set  $X$  of variables for two terms  $s$  and  $t$  is a set  $U$  of unifiers of  $s$  and  $t$  such that for every unifier  $\theta$  of  $s$  and  $t$  there exists a member  $\sigma \in U$  and a substitution  $\rho$  such that  $x\sigma\rho = x\theta$  for all  $x \in X$ . We let  $\text{CSU}_X(s, t)$  denote an arbitrary (preferably minimal) complete set of unifiers on  $X$  for  $s$  and  $t$ . We assume that all  $\sigma \in \text{CSU}_X(s, t)$  are idempotent on  $X$ —i.e.,  $x\sigma\sigma = x\sigma$  for all  $x \in X$ . The set  $X$  will consist of the free variables of the clauses in which  $s$  and  $t$  occur and will be left implicit.

**Semantics** A *type interpretation*  $\mathcal{J}_{\text{ty}} = (\mathcal{U}, \mathcal{J}_{\text{ty}})$  is defined as follows. The *universe*  $\mathcal{U}$  is a nonempty collection of nonempty sets, called *domains*. The function  $\mathcal{J}_{\text{ty}}$  associates a function  $\mathcal{J}_{\text{ty}}(\kappa) : \mathcal{U}^n \rightarrow \mathcal{U}$  with each  $n$ -ary type constructor  $\kappa$ , such that for all domains  $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{U}$ , the set  $\mathcal{J}_{\text{ty}}(\rightarrow)(\mathcal{D}_1, \mathcal{D}_2)$  is a subset of the function space from  $\mathcal{D}_1$  to  $\mathcal{D}_2$ . The semantics is *standard* if  $\mathcal{J}_{\text{ty}}(\rightarrow)(\mathcal{D}_1, \mathcal{D}_2)$  is the entire function space for all  $\mathcal{D}_1, \mathcal{D}_2$ .

A *type valuation*  $\xi$  is a function that maps every type variable to a domain. The *denotation* of a type for a type interpretation  $\mathcal{J}_{\text{ty}}$  and a type valuation  $\xi$  is defined by  $\llbracket \alpha \rrbracket_{\mathcal{J}_{\text{ty}}}^\xi = \xi(\alpha)$  and  $\llbracket \kappa(\bar{\tau}) \rrbracket_{\mathcal{J}_{\text{ty}}}^\xi = \mathcal{J}_{\text{ty}}(\kappa)(\llbracket \bar{\tau} \rrbracket_{\mathcal{J}_{\text{ty}}}^\xi)$ . We abuse notation by applying an operation on a tuple when it must be applied elementwise; thus,  $\llbracket \bar{\tau}_n \rrbracket_{\mathcal{J}_{\text{ty}}}^\xi$  stands for  $\llbracket \tau_1 \rrbracket_{\mathcal{J}_{\text{ty}}}^\xi, \dots, \llbracket \tau_n \rrbracket_{\mathcal{J}_{\text{ty}}}^\xi$ . A type valuation  $\xi$  can be extended to be a *valuation* by additionally assigning an element  $\xi(x) \in \llbracket \tau \rrbracket_{\mathcal{J}_{\text{ty}}}^\xi$  to each variable  $x : \tau$ . An *interpretation function*  $\mathcal{J}$  for a type interpretation  $\mathcal{J}_{\text{ty}}$  associates with each symbol  $f : \Pi \bar{\alpha}_m. \tau$  and domain tuple  $\bar{D}_m \in \mathcal{U}^m$  a value  $\mathcal{J}(f, \bar{D}_m) \in \llbracket \tau \rrbracket_{\mathcal{J}_{\text{ty}}}^\xi$ , where  $\xi$  is the type valuation that maps each  $\alpha_i$  to  $D_i$ .

The comprehension principle states that every function designated by a  $\lambda$ -expression is contained in the corresponding domain. Loosely following Fitting [33, Sect. 2.4], we initially allow  $\lambda$ -expressions to designate arbitrary elements of the domain, to be able to define the denotation of a term. We impose restrictions afterwards using the notion of a proper interpretation. A  *$\lambda$ -designation function*  $\mathcal{L}$  for a type interpretation  $\mathcal{J}_{\text{ty}}$  is a function that maps a valuation  $\xi$  and a  $\lambda$ -expression of type  $\tau$  to elements of  $\llbracket \tau \rrbracket_{\mathcal{J}_{\text{ty}}}^\xi$ . A type interpretation, an interpretation function, and a  $\lambda$ -designation function form an (*extensional*) *interpretation*  $\mathcal{J} = (\mathcal{J}_{\text{ty}}, \mathcal{J}, \mathcal{L})$ . For an interpretation  $\mathcal{J}$  and a valuation  $\xi$ , the denotation of a term is defined as  $\llbracket x \rrbracket_{\mathcal{J}}^\xi = \xi(x)$ ,  $\llbracket f(\bar{\tau}_m) \rrbracket_{\mathcal{J}}^\xi = \mathcal{J}(f, \llbracket \bar{\tau}_m \rrbracket_{\mathcal{J}_{\text{ty}}}^\xi)$ ,  $\llbracket s t \rrbracket_{\mathcal{J}}^\xi = \llbracket s \rrbracket_{\mathcal{J}}^\xi(\llbracket t \rrbracket_{\mathcal{J}}^\xi)$ , and  $\llbracket \lambda x. t \rrbracket_{\mathcal{J}}^\xi = \mathcal{L}(\xi, \lambda x. t)$ . For ground terms  $t$ , the denotation does not depend on the choice of the valuation  $\xi$ , which is why we sometimes write  $\llbracket t \rrbracket_{\mathcal{J}}$  for  $\llbracket t \rrbracket_{\mathcal{J}}^\xi$ .

An interpretation  $\mathcal{J}$  is *proper* if  $\llbracket \lambda x. t \rrbracket_{\mathcal{J}}^\xi(a) = \llbracket t \rrbracket_{\mathcal{J}}^{\xi[x \mapsto a]}$  for all  $\lambda$ -expressions  $\lambda x. t$  and all valuations  $\xi$ . If a type interpretation  $\mathcal{J}_{\text{ty}}$  and an interpretation function  $\mathcal{J}$  can be extended by a  $\lambda$ -designation function  $\mathcal{L}$  to a proper interpretation  $(\mathcal{J}_{\text{ty}}, \mathcal{J}, \mathcal{L})$ , then this  $\mathcal{L}$  is unique [33, Proposition 2.18]. Given an interpretation  $\mathcal{J}$  and a valuation  $\xi$ , an equation  $s \approx t$  is true if

$\llbracket s \rrbracket_J^\xi$  and  $\llbracket t \rrbracket_J^\xi$  are equal and it is false otherwise. A disequation  $s \not\approx t$  is true if  $s \approx t$  is false. A clause is true if at least one of its literals is true. A clause set is true if all its clauses are true. A proper interpretation  $J$  is a *model* of a clause set  $N$ , written  $J \models N$ , if  $N$  is true in  $J$  for all valuations  $\xi$ .

**Axiomatization of Booleans** Our clausal logic lacks a Boolean type, but it can easily be axiomatized as follows. We extend the signature with a nullary type constructor  $bool \in \Sigma_{\text{ty}}$  equipped with the proxy constants  $t, f : bool$ ,  $\text{not} : bool \rightarrow bool$ ,  $\text{and}, \text{or}, \text{impl}, \text{equiv} : bool \rightarrow bool \rightarrow bool$ ,  $\text{forall}, \text{exists} : \Pi\alpha. (\alpha \rightarrow bool) \rightarrow bool$ ,  $\text{eq} : \Pi\alpha. \alpha \rightarrow \alpha \rightarrow bool$ , and  $\text{choice} : \Pi\alpha. (\alpha \rightarrow bool) \rightarrow \alpha$ , characterized by the axioms

$$\begin{array}{lll}
t \not\approx f & \text{or } t x \approx t & \text{equiv } xy \approx \text{and } (\text{impl } xy) (\text{impl } yx) \\
x \approx t \vee x \approx f & \text{or } f x \approx x & \text{forall } \langle \alpha \rangle (\lambda x. t) \approx t \\
\text{not } t \approx f & \text{impl } t x \approx x & y \approx (\lambda x. t) \vee \text{forall } \langle \alpha \rangle y \approx f \\
\text{not } f \approx t & \text{impl } f x \approx t & \text{exists } \langle \alpha \rangle y \approx \text{not } (\text{forall } \langle \alpha \rangle (\lambda x. \text{not } (yx))) \\
\text{and } t x \approx x & x \not\approx y \vee \text{eq } \langle \alpha \rangle xy \approx t & y x \approx f \vee y (\text{choice } \langle \alpha \rangle y) \approx t \\
\text{and } f x \approx f & x \approx y \vee \text{eq } \langle \alpha \rangle xy \approx f &
\end{array}$$

This axiomatization of Booleans can be used in a prover to support full higher-order logic with or without Hilbert choice, corresponding to the TPTP THF format variants TH0 (monomorphic) [63] and TH1 (polymorphic) [43]. The prover’s clausifier would transform the outer first-order skeleton of a formula into a clause and use the axiomatized Booleans within the terms. It would also add the proxy axioms to the clausal problem.

### 3 The Calculus

Our *clausal  $\lambda$ -superposition calculus* is inspired by the extensional nonpurifying clausal  $\lambda$ -free higher-order superposition calculus described by Bentkamp et al. [13]. The text of this and the next section is partly based on that paper and the associated journal submission [12] (with Cruanes’s permission). The central idea is that superposition inferences are restricted to *unapplied* subterms occurring in the first-order outer skeleton of clauses—that is, outside  $\lambda$ -expressions and outside the arguments of applied variables. We call these “green subterms.” Thus,  $g \approx (\lambda x. f x x)$  cannot be used directly to rewrite  $g a$  to  $f a a$ , because  $g$  is applied in  $g a$ . A separate inference rule, ARGCONG, takes care of deriving  $g x \approx f x x$ , which can be oriented independently of its parent clause and used to rewrite  $g a$  or  $f a a$ .

A term (i.e., a  $\beta\eta$ -equivalence class)  $t$  is defined to be a *green subterm* of a term  $s$  if either  $s = t$  or  $s = f(\bar{t}) \bar{s}$  for some function symbol  $f$ , types  $\bar{t}$  and terms  $\bar{s}$ , where  $t$  is a green subterm of  $s_i$  for some  $i$ . In  $f(g a)(y b)(\lambda x. h c(g x))$ , the proper green subterms are  $a$ ,  $g a$ ,  $y b$ , and  $\lambda x. h c(g x)$ . The set of green positions of a term is defined analogously to the set of positions of a first-order term:  $\varepsilon$  is a green position of  $t$ , and if  $t = f(\bar{t}) \bar{s}$  and  $p$  is a green position of  $s_i$ , then  $i.p$  is a green position of  $t$ . We denote the green subterm of  $s$  at the green position  $p$  by  $s|_p$ . We write  $t = s \langle u \rangle_p$  to express that  $u$  is a green subterm of  $t$  at the green position  $p$  and call  $s \langle \rangle_p$  a *green context*; we omit the subscript  $p$  if there are no ambiguities.

Another key notion is that of a “fluid” term. A (not necessarily green) subterm  $t$  of  $s[t]$  is called *fluid* if (1)  $t \downarrow_{\beta\eta}$  is of the form  $y \bar{u}_n$ , where  $y$  is not bound in  $s[t]$  and  $n \geq 1$ , or (2)  $t \downarrow_{\beta\eta}$  is a  $\lambda$ -expression and there exists a substitution  $\sigma$  such that  $t\sigma \downarrow_{\beta\eta}$  is not a  $\lambda$ -expression (due to  $\eta$ -reduction). Case (2) can arise only if  $t$  contains an applied variable that is not bound in  $s[t]$ . Intuitively, fluid subterms are terms whose  $\eta$ -short  $\beta$ -normal form can change radically

as a result of instantiation. For example, applying  $\{z \mapsto \lambda x. x\}$  to the fluid term  $\lambda x. y a (z x)$  makes the  $\lambda$  vanish:  $(\lambda x. y a x) = y a$ . Similarly,  $(\lambda x. f (y x) x)\{y \mapsto \lambda x. a\} = (\lambda x. f a x) = f a$ .

### 3.1 The Core Inference Rules

The calculus is parameterized by a well-founded strict total order  $\succ$  on ground terms satisfying the following criteria, where  $\succeq$  denotes the reflexive closure of  $\succ$ :

- *green subterm property*:  $t \langle s \rangle \succeq s$ ;
- *compatibility with green contexts*:  $s' \succ s$  implies  $t \langle s' \rangle \succ t \langle s \rangle$ .

Two properties that are not required are *compatibility with  $\lambda$ -expressions* ( $s' \succ s$  implies  $(\lambda x. s') \succ (\lambda x. s)$ ) and *compatibility with arguments* ( $s' \succ s$  implies  $s' t \succ s t$ ). The latter would even be inconsistent with totality. To see why, consider the symbols  $c \succ b \succ a$  and the terms  $\lambda x. b$  and  $\lambda x. x$ . Owing to totality, one of the terms must be larger than the other, say,  $(\lambda x. b) \succ (\lambda x. x)$ . By compatibility with arguments, we get  $(\lambda x. b) c \succ (\lambda x. x) c$ , i.e.,  $b \succ c$ , a contradiction. A similar line of reasoning applies if  $(\lambda x. b) \prec (\lambda x. x)$ , using  $a$  instead of  $c$ .

For nonground terms, the only requirement on  $\succ$  is stability under grounding substitutions (i.e.,  $t \succ s$  implies  $t\theta \succ s\theta$  for all substitutions  $\theta$  grounding  $t$  and  $s$ ). The literal and clause orders are defined as multiset extensions in the standard way [6]. Despite their names, the term, literal, and clause orders need not be transitive on nonground entities.

The calculus is also parameterized by a relation  $\lesssim$  such that  $t \lesssim s$  implies  $t\theta \succeq s\theta$  for all grounding  $\theta$ . Normally,  $t \succeq s$  should imply  $t \lesssim s$ , but this is not a requirement. The relation  $\lesssim$  allows us to be more precise than the reflexive closure  $\succeq$  of  $\succ$ . For example, we must necessarily have  $y b \not\lesssim y a$  because  $y b \neq y a$  and  $y b \not\prec y a$  by stability under grounding substitutions (with  $\{y \mapsto \lambda x. c\}$ ). But we can have  $y b \lesssim y a$ . In contrast, with  $\lambda$ -free higher-order terms it is possible to have  $y b \succeq y a$ ; regardless of how  $y$  is instantiated,  $b$  and  $a$  will never vanish. In practice,  $\succ$  and  $\lesssim$  should be chosen so that they can compare as many pairs of terms as possible while being computable.

In addition to  $\succ$  and  $\lesssim$ , the calculus is parameterized by a selection function *sel* that maps each clause to a subclause consisting of negative literals. A literal  $L \langle y \rangle$  may not be selected if  $y \bar{u}_n$ , with  $n > 0$ , is a  $\lesssim$ -maximal term of the clause. A literal  $L$  is (*strictly*) *eligible* w.r.t. a substitution  $\sigma$  in  $C$  if it is selected in  $C$  or there are no selected literals in  $C$  and  $L\sigma$  is (*strictly*) maximal in  $C\sigma$ . If  $\sigma$  is the identity substitution, we leave it implicit.

A variable is *deep* in a clause  $C$  if it occurs inside a  $\lambda$ -expression or inside an argument of an applied variable; these cover all occurrences that may correspond to positions inside  $\lambda$ -expressions after applying a substitution. A variable that is not deep is said to be *shallow*.

We regard positive and negative superposition as two cases of a single rule

$$\frac{\overbrace{D' \vee t \approx t'}^D \quad \overbrace{C' \vee s \langle u \rangle \approx s'}^C}{(D' \vee C' \vee s \langle t' \rangle \approx s')\sigma} \text{SUP}$$

where  $\approx$  denotes either  $\approx$  or  $\not\approx$ . The following side conditions apply:

1.  $u$  is not a fluid subterm;
2.  $u$  is not a deep variable in  $C$ ;
3. *variable condition*: if  $u$  is a variable  $y$ , there must exist a grounding substitution  $\theta$  such that  $t\sigma\theta \succ t'\sigma\theta$  and  $C\sigma\theta \prec C''\sigma\theta$ , where  $C'' = C\{y \mapsto t'\}$ ;
4.  $\sigma \in \text{CSU}(t, u)$ ;      5.  $t\sigma \not\lesssim t'\sigma$ ;      6.  $s \langle u \rangle \sigma \not\lesssim s'\sigma$ ;      7.  $C\sigma \not\lesssim D\sigma$ ;

8.  $t \approx t'$  is strictly eligible in  $D$  w.r.t.  $\sigma$ ;
9.  $s\langle u \rangle \approx s'$  is eligible in  $C$  w.r.t.  $\sigma$ , and strictly eligible if it is positive.

There are four main differences with the statement of the standard superposition rule: Contexts  $s[\ ]$  are replaced by green contexts  $s\langle \rangle$ . The standard condition  $u \notin \mathcal{V}$  is generalized by conditions 2 and 3. Most general unifiers are replaced by complete sets of unifiers. And  $\not\approx$  is replaced by the more precise  $\not\approx$ .

The second rule is a variant of SUP that focuses on fluid subterms occurring in green contexts:

$$\frac{\overbrace{D' \vee t \approx t'}^D \quad \overbrace{C' \vee s\langle u \rangle \approx s'}^C}{(D' \vee C' \vee s\langle zt' \rangle \approx s')\sigma} \text{FLUIDSUP}$$

with the following side conditions, in addition to SUP's conditions 5 to 9:

1.  $u$  is either a fluid subterm or a deep variable in  $C$ ;
2.  $z$  is a fresh variable;
3.  $\sigma \in \text{CSU}(zt, u)$ ;
4.  $(zt')\sigma \neq (zt)\sigma$ .

The equality resolution and equality factoring rules are almost identical to their standard counterparts:

$$\frac{\overbrace{C' \vee u \not\approx u'}^C}{C'\sigma} \text{EQRES} \qquad \frac{\overbrace{C' \vee u' \approx v' \vee u \approx v}^C}{(C' \vee v \not\approx v' \vee u \approx v')\sigma} \text{EQFACT}$$

For EQRES:  $\sigma \in \text{CSU}(u, u')$  and  $u \not\approx u'$  is eligible in  $C$  w.r.t.  $\sigma$ . For EQFACT:  $\sigma \in \text{CSU}(u, u')$ ,  $u'\sigma \not\approx v'\sigma$ ,  $u\sigma \approx v\sigma$ , and  $u \approx v$  is eligible in  $C$  w.r.t.  $\sigma$ .

Argument congruence, a higher-order concern, is embodied by the rule

$$\frac{\overbrace{C' \vee s \approx s'}^C}{(C' \vee s \bar{x}_n \approx s' \bar{x}_n)\sigma} \text{ARGCONG}$$

where  $\sigma$  is the most general type substitution that ensures well-typedness of the conclusion. In particular, if the result type of  $s$  is not a type variable,  $\sigma$  is the identity substitution; and if the result type is a type variable, it is instantiated with  $\alpha_1 \rightarrow \dots \rightarrow \alpha_m \rightarrow \beta$ , where  $\bar{\alpha}_m$  and  $\beta$  are fresh. This yields infinitely many conclusions, one for each  $m$ . The literal  $s \approx s'$  must be strictly eligible in  $C$  w.r.t.  $\sigma$ , and  $\bar{x}_n$  is a nonempty tuple of distinct fresh variables.

The rules are complemented by the polymorphic functional extensionality axiom:

$$y(\text{diff}\langle \alpha, \beta \rangle yz) \not\approx z(\text{diff}\langle \alpha, \beta \rangle yz) \vee y \approx z \quad (\text{EXT})$$

From now on, we will omit the type arguments to  $\text{diff}$  since they can be inferred from the term arguments.

### 3.2 Rationale for the Rules

The calculus realizes the following division of labor: SUP and FLUIDSUP are responsible for green subterms, which are outside  $\lambda s$ , ARGCONG effectively gives access to the remaining positions outside  $\lambda s$ , and the extensionality axiom takes care of subterms inside  $\lambda s$ .

Because it gives rise to flex–flex pairs, which are unification constraints where both sides are applied variables, FLUIDSUP can be very prolific. With applied variables on both sides of its maximal literal, the extensionality axiom is another prime source of flex–flex pairs. Flex–flex pairs can also arise in the other rules (SUP, EQRES, and EQFACT). Due to order restrictions and fairness, we cannot postpone solving flex–flex pairs indefinitely. Thus, we cannot use Huet’s pre-unification procedure [39] and must instead choose a full unification procedure such as Jensen and Pietrzykowski’s [40], Snyder and Gallier’s [60], or the one by Vukmirović et al. [65]. On the positive side, optional inference rules can efficiently cover many cases where FLUIDSUP or the extensionality axiom would otherwise be needed (Sect. 5), and heuristics can help keep the explosion under control. Moreover, flex–flex pairs are not always as bad as their reputation; for example,  $y a b \stackrel{?}{=} z c d$  admits a most general unifier:  $\{y \mapsto \lambda w x. y' w x c d, z \mapsto y' a b\}$ . Full higher-order unification is currently being investigated by two of this article’s authors [65].

The calculus is a graceful generalization of standard superposition, except for the extensionality axiom. From  $g x \approx f x x$ , the axiom can be used to derive clauses such as  $(\lambda x. y x (g x)) \approx (\lambda x. y x (f x x))$ , which are useless if the problem is first-order. This could be avoided if we could find a way to make the positive literal  $y \approx z$  larger than the other literal, or to select  $y \approx z$  without losing refutational completeness. The literal  $y \approx z$  interacts only with green subterms of functional type, which do not arise in first-order clauses.

**Example 1** Prefix subterms such as  $g$  in the term  $g a$  are not green subterms and thus cannot be superposed into. ARGCONG gives us access to those positions. Consider the clauses  $g a \not\approx f a$  and  $g \approx f$ . An ARGCONG inference from  $g \approx f$  generates  $g x \approx f x$ . This clause can be used for a SUP inference into the first clause, yielding  $f a \not\approx f a$  and thus  $\perp$  by EQRES.

**Example 2** Applied variables give rise to subtle situations with no counterparts in first-order logic. Consider the clauses  $f a \approx c$  and  $h (y b) (y a) \not\approx h (g (f b)) (g c)$ , where  $f a \succ c$ . It is easy to see that the clause set is unsatisfiable, by grounding the second clause with  $\theta = \{y \mapsto \lambda x. g (f x)\}$ . However, to mimic the superposition inference that can be performed at the ground level, it is necessary to superpose at an imaginary position *below* the applied variable  $y$  and yet *above* its argument  $a$ , namely, into the subterm  $f a$  of  $g (f a) = (\lambda x. g (f x)) a = (y a)\theta$ . FLUIDSUP’s  $z$  variable effectively transforms  $f a \approx c$  into  $z (f a) \approx z c$ , whose left-hand side can be unified with  $y a$  by taking  $\{y \mapsto \lambda x. z (f x)\}$ . The resulting clause is  $h (z (f b)) (z c) \not\approx h (g (f b)) (g c)$ , from which  $\perp$  follows by EQRES.

**Example 3** The clause set consisting of  $f a \approx c$ ,  $f b \approx d$ , and  $g c \not\approx y a \vee g d \not\approx y b$  has a similar flavor. EQRES is applicable on either literal of the third clause, but the computed unifier,  $\{y \mapsto \lambda x. g c\}$  or  $\{y \mapsto \lambda x. g d\}$ , is not the right one. Again, we need FLUIDSUP.

**Example 4** Third-order clauses containing subterms of the form  $y (\lambda x. t)$  can be even more stupefying. The clause set containing  $f a \approx c$  and  $h (y (\lambda x. g (f x)) a) y \not\approx h (g c) (\lambda w x. w x)$  is unsatisfiable. To see why, apply  $\theta = \{y \mapsto \lambda w x. w x\}$  to the second clause, yielding  $h (g (f a)) (\lambda w x. w x) \not\approx h (g c) (\lambda w x. w x)$ . Let  $f a \succ c$ . A SUP inference is possible between the first clause and this ground instance of the second one. But at the nonground level, the subterm  $f a$  is not clearly localized:  $g (f a) = (\lambda x. g (f x)) a = (\lambda w x. w x) (\lambda x. g (f x)) a =$



$(y(\lambda x. g(f x)) a)\theta$ . The FLUIDSUP rule can cope with this. One of the unifiers of  $z(f a)$  and  $y(\lambda x. g(f x)) a$  will be  $\{y \mapsto \lambda w x. w x, z \mapsto g\}$ , yielding the clearly unsatisfiable clause  $h(g c)(\lambda w x. w x) \not\approx h(g c)(\lambda w x. w x)$ .

**Example 5** The FLUIDSUP rule is concerned not only with applied variables but also with  $\lambda$ -expressions that, after substitution, may be  $\eta$ -reduced to reveal new applied variables or green subterms. Consider the clauses  $g a \approx b$ ,  $h(\lambda y. x y g z) \approx c$ , and  $h(f b) \not\approx c$ . Applying  $\{x \mapsto \lambda y' w z'. f(w a)y'\}$  to the second clause yields  $h(\lambda y. (\lambda y' w z'. f(w a)y') y g z) \approx c$ , which  $\beta$ -reduces to  $h(\lambda y. f(g a)y) \approx c$  and  $\beta\eta$ -reduces to  $h(f(g a)) \approx c$ . A SUP inference is possible between the first clause set and this new ground clause, generating the clause  $h(f b) \approx c$ . By also considering  $\lambda$ -expressions, the FLUIDSUP rule is applicable at the nonground level to derive this clause.

### 3.3 Soundness

To show soundness of the inferences, we need the substitution lemma for our logic:

**Lemma 6 (Substitution lemma)** *Let  $\mathcal{J} = (\mathcal{J}_{\text{ty}}, \mathcal{J}, \mathcal{L})$  be a proper interpretation. Then*

$$\llbracket \tau \rho \rrbracket_{\mathcal{J}_{\text{ty}}}^{\xi} = \llbracket \tau \rrbracket_{\mathcal{J}_{\text{ty}}}^{\xi'} \quad \text{and} \quad \llbracket t \rho \rrbracket_{\mathcal{J}}^{\xi} = \llbracket t \rrbracket_{\mathcal{J}}^{\xi'}$$

for all terms  $t$ , all types  $\tau$ , and all substitutions  $\rho$ , where  $\xi'(\alpha) = \llbracket \alpha \rho \rrbracket_{\mathcal{J}_{\text{ty}}}^{\xi}$  for all type variables  $\alpha$  and  $\xi'(x) = \llbracket x \rho \rrbracket_{\mathcal{J}}^{\xi}$  for all term variables  $x$ .

*Proof* First, we prove that  $\llbracket \tau \rho \rrbracket_{\mathcal{J}_{\text{ty}}}^{\xi} = \llbracket \tau \rrbracket_{\mathcal{J}_{\text{ty}}}^{\xi'}$  by induction on the structure of  $\tau$ . If  $\tau = \alpha$  is a type variable,

$$\llbracket \alpha \rho \rrbracket_{\mathcal{J}_{\text{ty}}}^{\xi} = \xi'(\alpha) = \llbracket \alpha \rrbracket_{\mathcal{J}_{\text{ty}}}^{\xi'}$$

If  $\tau = \kappa(\bar{v})$  for some type constructor  $\kappa$  and types  $\bar{v}$ ,

$$\llbracket \kappa(\bar{v}) \rho \rrbracket_{\mathcal{J}_{\text{ty}}}^{\xi} = \mathcal{J}_{\text{ty}}(\kappa)(\llbracket \bar{v} \rho \rrbracket_{\mathcal{J}_{\text{ty}}}^{\xi}) \stackrel{\text{IH}}{=} \mathcal{J}_{\text{ty}}(\kappa)(\llbracket \bar{v} \rrbracket_{\mathcal{J}_{\text{ty}}}^{\xi'}) = \llbracket \kappa(\bar{v}) \rrbracket_{\mathcal{J}_{\text{ty}}}^{\xi'}$$

Next, we prove  $\llbracket t \rho \rrbracket_{\mathcal{J}}^{\xi} = \llbracket t \rrbracket_{\mathcal{J}}^{\xi'}$  by induction on the structure of a  $\lambda$ -term representative of  $t$ . If  $t = y$ , then by the definition of the denotation of a variable

$$\llbracket y \rho \rrbracket_{\mathcal{J}}^{\xi} = \xi'(y) = \llbracket y \rrbracket_{\mathcal{J}}^{\xi'}$$

If  $t = f(\bar{\tau})$ , then by the definition of the term denotation

$$\llbracket f(\bar{\tau}) \rho \rrbracket_{\mathcal{J}}^{\xi} = \mathcal{J}(f, \llbracket \bar{\tau} \rho \rrbracket_{\mathcal{J}_{\text{ty}}}^{\xi}) \stackrel{\text{IH}}{=} \mathcal{J}(f, \llbracket \bar{\tau} \rrbracket_{\mathcal{J}_{\text{ty}}}^{\xi'}) = \llbracket f(\bar{\tau}) \rrbracket_{\mathcal{J}}^{\xi'}$$

If  $t = uv$ , then by the definition of the term denotation

$$\llbracket (uv) \rho \rrbracket_{\mathcal{J}}^{\xi} = \llbracket u \rho \rrbracket_{\mathcal{J}}^{\xi} (\llbracket v \rho \rrbracket_{\mathcal{J}}^{\xi}) \stackrel{\text{IH}}{=} \llbracket u \rrbracket_{\mathcal{J}}^{\xi'} (\llbracket v \rrbracket_{\mathcal{J}}^{\xi'}) = \llbracket uv \rrbracket_{\mathcal{J}}^{\xi'}$$

If  $t$  is a  $\lambda$ -expression, let  $\rho'(z) = z$  and  $\rho'(x) = \rho(x)$  for  $x \neq z$ . Using properness of  $\mathcal{J}$  in the second and the last step, we have

$$\llbracket (\lambda z. u) \rho \rrbracket_{\mathcal{J}}^{\xi}(a) = \llbracket (\lambda z. u \rho') \rrbracket_{\mathcal{J}}^{\xi}(a) = \llbracket u \rho' \rrbracket_{\mathcal{J}}^{\xi[z \mapsto a]} \stackrel{\text{IH}}{=} \llbracket u \rrbracket_{\mathcal{J}}^{\xi'[z \mapsto a]} = \llbracket \lambda z. u \rrbracket_{\mathcal{J}}^{\xi'}(a) \quad \square$$

**Lemma 7** *If  $\mathcal{J} \models C$  for some interpretation  $\mathcal{J}$  and some clause  $C$ , then  $\mathcal{J} \models C\rho$  for all substitutions  $\rho$ .*

*Proof* We have to show that  $C$  is true in  $\mathcal{J}$  for all valuations  $\xi$ . Given a valuation  $\xi$ , define  $\xi'$  as in Lemma 6. Then, by Lemma 6, a literal in  $C\rho$  is true in  $\mathcal{J}$  for  $\xi$  if and only if the corresponding literal in  $C$  is true in  $\mathcal{J}$  for  $\xi'$ . There must be at least one such literal because  $\mathcal{J} \models C$  and hence  $C$  is in particular true in  $\mathcal{J}$  for  $\xi'$ . Therefore,  $C\rho$  is true in  $\mathcal{J}$  for  $\xi$ .

**Theorem 8 (Soundness)** *The inference rules SUP, FLUIDSUP, EQRES, EQFACT, and ARGCONG are sound (even without the variable condition and the side conditions on fluidity, deep variables, order, and eligibility).*

*Proof* We fix an inference and an interpretation  $\mathcal{J}$  that is a model of the premises. We need to show that it is also a model of the conclusion.

From the definition of the denotation of a term, it is obvious that congruence holds in our logic, at least for subterms that are not inside a  $\lambda$ -expression. In particular, it holds for green subterms and for the left subterm  $t$  of an application  $ts$ .

By Lemma 7,  $\mathcal{J}$  is a model of the  $\sigma$ -instances of the premises as well, where  $\sigma$  is the substitution used for the inference. Let  $\xi$  be a valuation. By making case distinctions on the truth under  $\mathcal{J}, \xi$  of the literals of the  $\sigma$ -instances of the premises, using the conditions that  $\sigma$  is a unifier, and applying congruence, it follows that the conclusion is true under  $\mathcal{J}, \xi$ . Hence,  $\mathcal{J}$  is a model of the conclusion.  $\square$

As in the  $\lambda$ -free higher-order logic of Bentkamp et al. [12], skolemization is unsound in our logic. As a consequence, the axiom (EXT) does not hold in all interpretations, but the axiom is consistent with our logic, i.e., there exist models of (EXT).

### 3.4 Redundancy Criterion

A redundant (or composite) clause is usually defined as a clause whose ground instances are entailed by smaller ( $\prec$ ) ground instances of existing clauses. This would be too strong for our calculus, as it would make most clauses produced by ARGCONG redundant. The solution is to base the redundancy criterion on a weaker ground logic—ground monomorphic first-order logic—in which argument congruence and extensionality do not hold. The resulting notion of redundancy gracefully generalizes the standard first-order notion.

We employ an encoding  $\mathcal{F}$  to translate ground higher-order terms into ground first-order terms.  $\mathcal{F}$  indexes each symbol occurrence with the type arguments and the number of term arguments. For example,  $\mathcal{F}(f \mathbf{a}) = f_1(\mathbf{a}_0)$  and  $\mathcal{F}(g(\kappa)) = g_0^\kappa$ . In addition,  $\mathcal{F}$  conceals  $\lambda$ -expressions by replacing them with fresh symbols. These measures effectively disable argument congruence and extensionality. For example, the clause sets  $\{g \approx f, g \mathbf{a} \not\approx f \mathbf{a}\}$  and  $\{b \approx a, (\lambda x. b) \not\approx (\lambda x. a)\}$  are unsatisfiable in higher-order logic, but the encoded clause sets  $\{g_0 \approx f_0, g_1(\mathbf{a}_0) \not\approx f_1(\mathbf{a}_0)\}$  and  $\{b_0 \approx a_0, c_0 \not\approx d_0\}$  are satisfiable in first-order logic.

Given a higher-order signature  $(\Sigma_{\text{ty}}, \mathcal{V}_{\text{ty}}, \Sigma, \mathcal{V})$ , we define a ground first-order signature  $(\Sigma_{\text{ty}}, \{\}, \Sigma_{\text{GF}}, \{\})$  as follows. The type constructors  $\Sigma_{\text{ty}}$  are the same in both signatures, but  $\rightarrow$  is uninterpreted in first-order logic. For each ground instance  $f(\bar{v}) : \tau_1 \rightarrow \dots \rightarrow \tau_n \rightarrow \tau$  of a symbol  $f \in \Sigma$ , we introduce a first-order symbol  $f_j^{\bar{v}} \in \Sigma_{\text{GF}}$  with argument types  $\bar{\tau}_j$  and result type  $\tau_{j+1} \rightarrow \dots \rightarrow \tau_n \rightarrow \tau$ , for each  $j$ . Moreover, for each ground term  $\lambda x. t$ , we introduce a symbol  $\text{lam}_{\lambda x. t} \in \Sigma_{\text{GF}}$  of the same type.

Thus, we consider three layers of logics: the higher-order layer H over a given signature  $(\Sigma_{\text{ty}}, \mathcal{V}_{\text{ty}}, \Sigma, \mathcal{V})$ , the ground higher-order layer GH over the signature  $(\Sigma_{\text{ty}}, \{\}, \Sigma, \{\})$ , and the ground monomorphic first-order layer GF over the signature  $(\Sigma_{\text{ty}}, \{\}, \Sigma_{\text{GF}}, \{\})$  defined above. We use  $\mathcal{T}_H$ ,  $\mathcal{T}_{\text{GH}}$ , and  $\mathcal{T}_{\text{GF}}$  to denote the respective sets of terms,  $\mathcal{T}y_H$ ,  $\mathcal{T}y_{\text{GH}}$ , and  $\mathcal{T}y_{\text{GF}}$  to

denote the respective sets of types and  $C_H$ ,  $C_{GH}$ , and  $C_{GF}$  to denote the respective sets of clauses. Each of the three layers has an entailment relation  $\models$ . A clause set  $N_1$  entails a clause set  $N_2$ , denoted  $N_1 \models N_2$ , if any model of  $N_1$  is also a model of  $N_2$ . For H and GH, we use higher-order models; for GF, we use first-order models. This machinery may seem excessive, but it is essential to define redundancy of clauses and inferences properly, and it will play an important role in the refutational completeness proof (Sect. 4).

The three layers are connected by two functions  $\mathcal{G}$  and  $\mathcal{F}$ . The grounding function  $\mathcal{G}$  maps terms from  $\mathcal{T}_H$  to the set of their ground instances in  $\mathcal{T}_{GH}$  and clauses from  $C_H$  to the set of their ground instances in  $C_{GH}$ . The encoding  $\mathcal{F} : \mathcal{T}_{GH} \rightarrow \mathcal{T}_{GF}$  is recursively defined as

$$\mathcal{F}(\lambda x.t) = \text{lam}_{\lambda x.t} \qquad \mathcal{F}(f(\bar{v})\bar{s}_j) = f_j^{\bar{v}}(\mathcal{F}(\bar{s}_j))$$

using  $\eta$ -short  $\beta$ -normal representatives of terms. The encoding  $\mathcal{F}$  is extended to map GH clauses to GF clauses in the obvious way. The mapping  $\mathcal{F}$  is clearly bijective. Using the inverse mapping, the order  $\succ$  can be transferred from  $\mathcal{T}_{GH}$  to  $\mathcal{T}_{GF}$  and from  $C_{GH}$  to  $C_{GF}$  by defining  $t \succ s$  as  $\mathcal{F}^{-1}(t) \succ \mathcal{F}^{-1}(s)$  and  $C \succ D$  as  $\mathcal{F}^{-1}(C) \succ \mathcal{F}^{-1}(D)$ . The property that  $\succ$  on clauses is the multiset extension of  $\succ$  on literals, which in turn is the multiset extension of  $\succ$  on terms, is maintained because  $\mathcal{F}^{-1}$  maps the multiset representations elementwise.

A key property of  $\mathcal{F}$  is that green subterms in GH correspond to subterms in GF. This allows us to show that well-foundedness, totality on ground terms, compatibility with contexts, and the subterm property hold for  $\succ$  on GF terms. Moreover, the subterms considered by SUP and FLUIDSUP include all the subterms exposed to the redundancy criterion.

**Lemma 9** *Let  $s, t \in \mathcal{T}_{GH}$ . We have  $\mathcal{F}(t \langle s \rangle_p) = \mathcal{F}(t)[\mathcal{F}(s)]_p$ . In other words,  $s$  is a green subterm of  $t$  at position  $p$  if and only if  $\mathcal{F}(s)$  is a subterm of  $\mathcal{F}(t)$  at position  $p$ .*

*Proof* Analogous to Lemma 3.13 of Bentkamp et al. [12].

**Lemma 10** *Well-foundedness, totality, compatibility with contexts, and the subterm property hold for  $\succ$  in  $\mathcal{T}_{GF}$ .*

*Proof* Analogous to Lemma 3.15 of Bentkamp et al. [12].

The saturation procedures of superposition provers aggressively delete clauses that are strictly subsumed by other clauses. A clause  $C$  *subsumes*  $D$  if there exists a substitution  $\sigma$  such that  $C\sigma \subseteq D$ . A clause  $C$  *strictly subsumes*  $D$  if  $C$  subsumes  $D$  but  $D$  does not subsume  $C$ . For example,  $x \approx c$  strictly subsumes both  $a \approx c$  and  $b \not\approx a \vee x \approx c$ . The proof of refutational completeness of resolution and superposition provers relies on the well-foundedness of the strict subsumption relation [57, Sect. 7]. Unfortunately, this property does not hold for higher-order logic, where  $f x x \approx c$  is strictly subsumed by  $f(x a)(x b) \approx c$ , which is strictly subsumed by  $f(x a a')(x b b') \approx c$ , and so on. To prevent such infinite chains, we use a well-founded partial order  $\sqsupseteq$  on  $C_H$ . We can define  $\sqsupseteq$  as  $\succsim \cap \succ_{\text{size}}$ , where  $\succsim$  stands for “subsumed by” and  $D \succ_{\text{size}} C$  if either  $\text{size}(D) > \text{size}(C)$  or  $\text{size}(D) = \text{size}(C)$  and  $D$  contains fewer distinct variables than  $C$ ; the *size* function is some notion of syntactic size, such as the number of constants and variables contained in a clause. This yields for instance  $a \approx c \sqsupseteq x \approx c$  and  $f(x a a) \approx c \sqsupseteq f(y a) \approx c$ . To justify the deletion of subsumed clauses, we set up our redundancy criterion to cover subsumption, following Waldmann et al. [67].

We define the sets of redundant clauses w.r.t. a given clause set as follows:

- Given  $C \in C_{GF}$  and  $N \subseteq C_{GF}$ , let  $C \in \text{GFRed}_C(N)$  if  $\{D \in N \mid D \prec C\} \models C$ .
- Given  $C \in C_{GH}$  and  $N \subseteq C_{GH}$ , let  $C \in \text{GHRed}_C(N)$  if  $\mathcal{F}(C) \in \text{GFRed}_C(\mathcal{F}(N))$ .

- Given  $C \in \mathcal{C}_H$  and  $N \subseteq \mathcal{C}_H$ , let  $C \in \mathit{HRed}_C(N)$  if for every  $D \in \mathcal{G}(C)$ , we have  $D \in \mathit{GHRed}_C(\mathcal{G}(N))$  or there exists  $C' \in N$  such that  $C \sqsupset C'$  and  $D \in \mathcal{G}(C)$ .

Along with the three layers of logics, we consider three inference systems:  $\mathit{HInf}$ ,  $\mathit{GHInf}$ , and  $\mathit{GFInf}$ .  $\mathit{HInf}$  is the inference system described in Sect. 3.1. For uniformity, we regard the extensionality axiom as a premise-free inference rule EXT whose conclusion is axiom (EXT).  $\mathit{GHInf}$  consists of all SUP, EQRES, and EQFACT inferences from  $\mathit{HInf}$  whose premises and conclusion are ground, a premise-free rule GEXT whose infinitely many conclusions are the ground instances of (EXT), and the following ground variant of ARGCONG:

$$\frac{C' \vee s \approx s'}{C' \vee s \bar{u}_n \approx s' \bar{u}_n} \text{GARGCONG}$$

where  $s \approx s'$  is strictly eligible in  $C' \vee s \approx s'$  and  $\bar{u}_n$  is a nonempty tuple of ground terms.  $\mathit{GFInf}$  contains all SUP, EQRES, and EQFACT inferences from  $\mathit{GHInf}$  translated by  $\mathcal{F}$ . It coincides with standard first-order superposition. Given a SUP, EQRES, or EQFACT inference  $\iota \in \mathit{GHInf}$ , let  $\mathcal{F}(\iota)$  denote the corresponding inference in  $\mathit{GFInf}$ .

Given an inference  $\iota$ , we write  $\mathit{prems}(\iota)$  for the tuple of premises,  $\mathit{mprem}(\iota)$  for the main (i.e., rightmost) premise, and  $\mathit{concl}(\iota)$  for the conclusion.

Each of the three inference systems is parameterized by a selection function  $\mathit{sel}$ . Occasionally, we will make this dependency explicit, writing  $\mathit{HInf}^{\mathit{sel}}$  for  $\mathit{HInf}$  and similarly for  $\mathit{GHInf}$  and  $\mathit{GFInf}$ . For each selection function  $\mathit{sel}$  on  $\mathcal{C}_{GH}$ , via the bijection  $\mathcal{F}$ , we can obtain a corresponding selection function on  $\mathcal{C}_{GF}$ , which we denote  $\mathcal{F}(\mathit{sel})$ . There is, however, no general way to derive the right selection function on  $\mathcal{C}_{GH}$  from a selection function on  $\mathcal{C}_H$ . In the refutational completeness proof, given a saturated clause set  $N \subseteq \mathcal{C}_H$  and a selection function on  $\mathcal{C}_H$ , we need a selection function on  $\mathcal{C}_{GH}$  such that for each clause  $C \in \mathcal{G}(N)$  there exists a clause  $D \in N$  with  $C \in \mathcal{G}(D)$  and corresponding selected literals. Since the saturated clause set  $N$  is not known during a derivation, our redundancy criterion must not depend on it. Therefore, we consider all selection functions on  $\mathcal{C}_{GH}$  such that for each clause in  $C \in \mathcal{C}_{GH}$ , there exists a clause  $D \in \mathcal{C}_H$  with  $C \in \mathcal{G}(D)$  and corresponding selected literals. Given a selection function  $\mathit{sel}$  on  $\mathcal{C}_H$ , let  $\mathcal{G}(\mathit{sel})$  denote the set of such selection functions on  $\mathcal{C}_{GH}$ .

Given a selection function  $\mathit{sel}$  on  $\mathcal{C}_H$ , a selection function  $\mathit{gsel} \in \mathcal{G}(\mathit{sel})$ , and an inference  $\iota \in \mathit{HInf}^{\mathit{sel}}$ , we define the set  $\mathcal{G}^{\mathit{gsel}}(\iota)$  of ground instances of  $\iota$  to be all inferences  $\iota' \in \mathit{GHInf}^{\mathit{gsel}}$  such that  $\mathit{prems}(\iota') = \mathit{prems}(\iota)\theta$  and  $\mathit{concl}(\iota') = \mathit{concl}(\iota)\theta$  for some grounding substitution  $\theta$ . This will map SUP and FLUIDSUP to SUP, EQFACT to EQFACT, EQRES to EQRES, EXT to GEXT, and ARGCONG to GARGCONG inferences, but it is also possible that  $\mathcal{G}^{\mathit{gsel}}(\iota)$  is the empty set for some inferences  $\iota$ .

We define the sets of redundant inferences w.r.t. a given clause set as follows:

- Given  $\iota \in \mathit{GFInf}^{\mathit{gsel}}$  and  $N \subseteq \mathcal{C}_{GF}$ , let  $\iota \in \mathit{GFRed}_1^{\mathit{gsel}}(N)$  if  $\mathit{prems}(\iota) \cap \mathit{GFRed}_C(N) \neq \emptyset$  or  $\{D \in N \mid D \prec \mathit{mprem}(\iota)\} \models \mathit{concl}(\iota)$ .
- Given  $\iota \in \mathit{GHInf}^{\mathit{gsel}}$  and  $N \subseteq \mathcal{C}_{GH}$ , let  $\iota \in \mathit{GHRed}_1^{\mathit{gsel}}(N)$  if
  - $\iota$  is not a GARGCONG or GEXT inference and  $\mathcal{F}(\iota) \in \mathit{GFRed}_1^{\mathcal{F}(\mathit{gsel})}(\mathcal{F}(N))$ ; or
  - $\iota$  is a GARGCONG or GEXT inference and  $\mathit{concl}(\iota) \in N \cup \mathit{GHRed}_C(N)$ .
- Given  $\iota \in \mathit{HInf}^{\mathit{sel}}$  and  $N \subseteq \mathcal{C}_H$ , let  $\iota \in \mathit{HRed}_1(N)$  if  $\mathcal{G}^{\mathit{gsel}}(\iota) \subseteq \mathit{GHRed}_1(\mathcal{G}(N))$  for all  $\mathit{gsel} \in \mathcal{G}(\mathit{sel})$ .

Occasionally, we omit the selection function in the notation when it is irrelevant. A clause set  $N$  is *saturated* w.r.t. an inference system and a redundancy criterion ( $\mathit{Red}_1, \mathit{Red}_C$ ) if every inference from clauses in  $N$  is in  $\mathit{Red}_1(N)$ .

### 3.5 A Derived Term Order

We stated some requirements on  $\succ$  and  $\succsim$  in Sect. 3.1 but have not shown how to fulfill them. To derive a suitable order  $\succ$ , we propose to encode  $\eta$ -short  $\beta$ -normal forms into untyped  $\lambda$ -free higher-order terms and apply an order  $\succ_{\mathcal{M}}$  such as the  $\lambda$ -free Knuth–Bendix order [9], the  $\lambda$ -free lexicographic path order [23], or the embedding path order [10].

The encoding, denoted by  $O$ , translates  $\lambda x:\tau. t$  to  $\text{lam } O(\tau) O(t)$  and uses De Bruijn [27] symbols  $\text{db}_i$  to represent bound variables  $x$ . It replaces fluid terms  $t$  by fresh variables  $z_t$  and maps type arguments to term arguments, while erasing any other type information; thus,  $O(\lambda x:\kappa. \lambda y:\kappa. x) = \text{lam } \kappa (\text{lam } \kappa (\text{db}_1 \kappa))$  and  $O(f\langle\kappa\rangle(y a)) = f \kappa z_{ya}$ . The use of De Bruijn indices and the monolithic encoding of fluid terms ensure stability under  $\alpha$ -renaming and substitution. We call the resulting order  $\succ_{\lambda}$ . For  $\succsim$ , we can use  $\succeq_{\lambda}$ .

Given a signature  $(\Sigma_{\text{ty}}, \mathcal{V}_{\text{ty}}, \Sigma, \mathcal{V})$ ,  $O$  encodes types and terms as terms over the untyped  $\lambda$ -free signature  $(\Sigma_{\text{ty}} \uplus \Sigma \uplus \{\text{lam}\} \uplus \{\text{db}_i \mid i \in \mathbb{N}\}, \{z_t \mid t \text{ is a term}\})$ . The auxiliary function  $\mathcal{B}_x(t)$  replaces each free occurrence of the variable  $x$  by the symbol  $\text{db}_i$ , where  $i$  is the number of  $\lambda$ -expressions surrounding the variable occurrence; for example,  $\mathcal{B}_x(f x(\lambda w. g w x)(\lambda x. x)) = f \text{db}_0(\lambda w. g w \text{db}_1)(\lambda x. x)$ . The type-to-term version of  $O$  is defined by  $O(\alpha) = \alpha$  and  $O(\kappa(\bar{\tau})) = \kappa(O(\bar{\tau}))$ . The term-to-term version is defined by

$$O(t) = \begin{cases} z_t & \text{if } t = x \text{ or } t \text{ is fluid} \\ \text{lam } O(\tau) O(\mathcal{B}_x(u)) & \text{if } t = (\lambda x:\tau. u) \text{ and } t \text{ is not fluid} \\ f O(\bar{\tau}) O(\bar{u}) & \text{if } t = f\langle\bar{\tau}\rangle\bar{u} \end{cases}$$

**Lemma 11** *Let  $\succ_{\mathcal{M}}$  be a strict partial order on  $\lambda$ -free terms and  $\succ_{\lambda}$  the induced order on  $\beta\eta$ -equivalence classes. If the restriction of  $\succ_{\mathcal{M}}$  to ground terms enjoys well-foundedness, totality, the green subterm property, and compatibility with green contexts, the restriction of  $\succ_{\lambda}$  to ground terms enjoys the same properties.*

*Proof* Transitivity and irreflexivity of  $\succ_{\mathcal{M}}$  imply transitivity and irreflexivity of  $\succ_{\lambda}$ .

**WELL-FOUNDEDNESS:** If there existed an infinite chain  $t_1 \succ_{\lambda} t_2 \succ_{\lambda} \dots$  of ground terms, there would also be the chain  $O(t_1) \succ_{\mathcal{M}} O(t_2) \succ_{\mathcal{M}} \dots$ , contradicting the well-foundedness of  $\succ_{\mathcal{M}}$  on ground  $\lambda$ -free terms.

**TOTALITY:** By ground totality of  $\succ_{\mathcal{M}}$ , for any ground terms  $t$  and  $s$  we have  $O(t) \succ_{\mathcal{M}} O(s)$ ,  $O(t) \prec_{\mathcal{M}} O(s)$ , or  $O(t) = O(s)$ . In the first two cases, it follows that  $t \succ_{\lambda} s$  or  $t \prec_{\lambda} s$ . In the last case, it follows that  $t = s$  because  $O$  is clearly injective.

**GREEN SUBTERM PROPERTY:** Let  $s$  be a term. We show that  $s \succeq_{\lambda} s|_p$  by induction on  $p$ , where  $s|_p$  denotes the green subterm at position  $p$ . If  $p = \varepsilon$ , this is trivial. If  $p = p'.i$ , we have  $s \succeq_{\lambda} s|_{p'}$  by the induction hypothesis. Hence, it suffices to show that  $s|_{p'} \succeq_{\lambda} s|_{p'.i}$ . From the existence of the position  $p'.i$ , we know that  $s|_{p'}$  must be of the form  $s|_{p'} = f\langle\bar{\tau}\rangle\bar{u}$ . Then  $s|_{p'.i} = u_i$ . The encoding yields  $O(s|_{p'}) = f O(\bar{\tau}) O(\bar{u})$  and hence  $O(s|_{p'}) \succeq_{\mathcal{M}} O(s|_{p'.i})$  by the ground green subterm property of  $\succ_{\mathcal{M}}$ . Hence,  $s|_{p'} \succeq_{\lambda} s|_{p'.i}$  and thus  $s \succeq_{\lambda} s|_p$ .

**COMPATIBILITY WITH GREEN CONTEXTS:** By induction on the depth of the context, it suffices to show that  $t \succ_{\lambda} s$  implies  $f\langle\bar{\tau}\rangle\bar{u} t \bar{v} \succ_{\lambda} f\langle\bar{\tau}\rangle\bar{u} s \bar{v}$  for all  $t, s, f, \bar{\tau}, \bar{u}$ , and  $\bar{v}$ . This amounts to showing that  $O(t) \succ_{\mathcal{M}} O(s)$  implies  $O(f\langle\bar{\tau}\rangle\bar{u} t \bar{v}) = f O(\bar{\tau}) O(\bar{u}) O(t) O(\bar{v}) \succ_{\mathcal{M}} f O(\bar{\tau}) O(\bar{u}) O(s) O(\bar{v}) = O(f\langle\bar{\tau}\rangle\bar{u} s \bar{v})$ , which follows directly from ground compatibility of  $\succ_{\mathcal{M}}$  with green contexts and the induction hypothesis.  $\square$

**Lemma 12** *Let  $\succ_{\lambda\mathcal{F}}$  be a strict partial order on  $\lambda$ -free terms. If  $\succ_{\lambda\mathcal{F}}$  is stable under grounding substitutions (w.r.t.  $\lambda$ -free terms), the induced order  $\succ_{\lambda}$  is stable under grounding substitutions (w.r.t.  $\beta\eta$ -equivalence classes).*

*Proof* First, we extend  $O$  so that  $O(\text{db}_i) = \text{db}_i$  for all  $i$ . Let  $\theta$  be a substitution. Given a finite set  $T$  of  $\lambda$ -terms, we define a  $\lambda$ -free substitution  $\rho$  such that  $z_u\rho = O(u\theta)$  for all subterms  $u$  of  $t \in T$  and otherwise  $x\rho = x$ . Then  $O(t)\rho = O(t\theta)$  for all  $t \in T$  by induction on the definition of the encoding. If  $t = x$  or if  $t$  is fluid,  $O(t)\rho = z_t\rho = O(t\theta)$ . If  $t = f(\bar{\tau})\bar{u}$ , then  $O(t)\rho = f(O(\bar{\tau})\rho)(O(\bar{u})\rho) \stackrel{\text{IH}}{=} f O(\bar{\tau}\theta) O(\bar{u}\theta) = O(f(\bar{\tau}\theta)(\bar{u}\theta)) = O(t\theta)$ . If  $t = (\lambda x : \tau. u)$  and  $t$  is not fluid, then  $O(t)\rho = \text{lam}(O(\tau)\rho)(O(\mathcal{B}_x(u))\rho) \stackrel{\text{IH}}{=} \text{lam} O(\tau\theta) O(\mathcal{B}_x(u)\theta) = \text{lam} O(\tau\theta) O(\mathcal{B}_x(u)\theta[x \mapsto x]) = O(\lambda x : \tau\theta. u\theta[x \mapsto x]) = O((\lambda x : \tau. u)\theta) = O(t\theta)$ .

Assume  $t \succ_{\lambda} s$  for some terms  $t$  and  $s$  and hence  $O(t) \succ_{\lambda\mathcal{F}} O(s)$ . Assume that  $\theta$  grounds  $t$  and  $s$ . Then the substitution  $\rho$  associated with  $\{t, s\}$  grounds  $O(t)$  and  $O(s)$ . By stability of  $\succ_{\lambda\mathcal{F}}$  under grounding substitutions,  $O(t)\rho \succ_{\lambda\mathcal{F}} O(s)\rho$ . By construction of  $\rho$ , it follows that  $O(t\theta) \succ_{\lambda\mathcal{F}} O(s\theta)$  and hence  $t\theta \succ_{\lambda} s\theta$ .  $\square$

## 4 Refutational Completeness

Besides soundness, the most important property of the clausal  $\lambda$ -superposition calculus introduced in Sect. 3 is refutational completeness.

### 4.1 Outline of the Proof

The proof proceeds in three steps, corresponding to the three layers GF, GH, and H introduced in Sect. 3.4:

1. We use Bachmair and Ganzinger’s work on the refutational completeness of standard (first-order) superposition [6] to prove static refutational completeness of  $GFInf$ .
2. From the first-order model constructed in Bachmair and Ganzinger’s proof, we derive a clausal higher-order model and thus prove static refutational completeness of  $GHInf$ .
3. We use the saturation framework by Waldmann et al. [67] to lift the static refutational completeness of  $GHInf$  to static and dynamic refutational completeness of  $HInf$ .

In the first step, since the inference system  $GFInf$  is standard ground superposition, we can make use of Bachmair and Ganzinger’s results. Given a saturated clause set  $N \subseteq C_{GF}$  with  $\perp \notin N$ , Bachmair and Ganzinger prove refutational completeness by constructing a term rewriting system  $R_N$  and showing that it can be viewed as an interpretation that is a model of  $N$ . This first step deals exclusively with ground first-order clauses.

In the second step, we derive refutational completeness of  $GHInf$ . Given a saturated clause set  $N \subseteq C_{GH}$  with  $\perp \notin N$ , we use the first-order model  $R_{\mathcal{F}(N)}$  of  $\mathcal{F}(N)$  constructed in the first step to derive a clausal higher-order interpretation that is a model of  $N$ . Under the encoding  $\mathcal{F}$ , occurrences of the same symbol with different numbers of arguments are regarded as different symbols—e.g.,  $\mathcal{F}(f) = f_0$  and  $\mathcal{F}(f a) = f_1(a_0)$ . All  $\lambda$ -expressions  $\lambda x. t$  are regarded as uninterpreted symbols  $\text{lam}_{\lambda x. t}$ . The difficulty is to construct a higher-order interpretation that merges the first-order denotations of all  $f_i$  into a single higher-order denotation of  $f$  and to show that the symbols  $\text{lam}_{\lambda x. t}$  behave like  $\lambda x. t$ . This step relies on saturation w.r.t. the GARGCONG rule—which connects a term of functional type with its value when applied to an argument  $x$ —and on the presence of the extensionality rule GEXT.

In the third step, we employ the saturation framework by Waldmann et al. [67] to prove refutational completeness of  $HInf$ . The main proof obligation the framework leaves to us is that there exist nonground inferences in  $HInf$  corresponding to all nonredundant inferences in  $GHInf$ . We face two specifically higher-order difficulties. First, in standard superposition, we can avoid SUP inferences into variables  $x$  by exploiting the clause order's compatibility with contexts: If  $t' \prec t$ , we have  $C\{x \mapsto t'\} \prec C\{x \mapsto t\}$ , which allows us to show that SUP inferences into variables are redundant. This technique fails for higher-order variables  $x$  that occur applied in  $C$ , because the order lacks compatibility with arguments. This is why our SUP rule must perform some inferences into variables. The other difficulty also concerns applied variables. We must show that any nonredundant SUP inference on layer GH into a position corresponding to a fluid term or a deep variable on layer H can be lifted to a FLUIDSUP inference. This involves showing that the  $z$  variable in FLUIDSUP can represent arbitrary contexts around a term  $t$ .

For the entire proof of refutational completeness,  $\beta\eta$ -normalization is the proverbial dog that did not bark. On layer GH, the rules SUP, EQRES, and EQFACT preserve  $\eta$ -short  $\beta$ -normal form, and so does first-order term rewriting. Thus, we can completely ignore  $\rightarrow_\beta$  and  $\rightarrow_\eta$ . On layer H, instantiation can cause  $\beta$ - and  $\eta$ -reduction, but this poses no difficulties thanks to the clause order's stability under grounding substitutions.

## 4.2 The Ground First-Order Layer

We use Bachmair and Ganzinger's results on standard superposition [6] to prove refutational completeness of GF. In the subsequent steps, we will also make use of specific properties of the model Bachmair and Ganzinger construct. The basis of Bachmair and Ganzinger's proof is that a term rewriting system  $R$  defines an interpretation  $\mathcal{T}_{GF}/R$  such that for every ground equation  $s \approx t$ , we have  $\mathcal{T}_{GF}/R \models s \approx t$  if and only if  $s \leftrightarrow_R^* t$ . Formally,  $\mathcal{T}_{GF}/R$  denotes the monomorphic first-order interpretation whose universes  $\mathcal{U}_\tau$  consist of the  $R$ -equivalence classes over  $\mathcal{T}_{GF}$  containing terms of type  $\tau$ . The interpretation  $\mathcal{T}_{GF}/R$  is term-generated—that is, for every element  $a$  of the universe of this interpretation and for any valuation  $\xi$ , there exists a ground term  $t$  such that  $\llbracket t \rrbracket_{\mathcal{T}_{GF}/R}^\xi = a$ . To lighten notation, we will write  $R$  to refer to both the term rewriting system  $R$  and the interpretation  $\mathcal{T}_{GF}/R$ .

The term rewriting system is constructed as follows. Let  $N \subseteq C_{GF}$ . We first define sets of rewrite rules  $E_N^C$  and  $R_N^C$  for all  $C \in N$  by induction on the clause order. Assume that  $E_N^D$  has already been defined for all  $D \in N$  such that  $D \prec C$ . Then  $R_N^C = \bigcup_{D \prec C} E_N^D$ . Let  $E_N^C = \{s \rightarrow t\}$  if the following conditions are met:

- (a)  $C = C' \vee s \approx t$ ;
- (b)  $s \approx t$  is strictly maximal in  $C$ ;
- (c)  $s \succ t$ ;
- (d)  $C'$  is false in  $R_N^C$ ;
- (e)  $s$  is irreducible w.r.t.  $R_N^C$ .

Then  $C$  is said to *produce*  $s \rightarrow t$ . Otherwise,  $E_N^C = \emptyset$ . Finally,  $R_N = \bigcup_D E_N^D$ . Based on Bachmair and Ganzinger's work, Bentkamp et al. [12, Lemma 4.2 and Theorem 4.3] prove the following properties of  $R_N$ :

**Lemma 13** *Let  $\perp \notin N$  and  $N \subseteq C_{GF}$  be saturated w.r.t.  $GFInf$  and  $GFRed_1$ . If  $C = C' \vee s \approx t \in N$  produces  $s \rightarrow t$ , then  $s \approx t$  is strictly eligible in  $C$  and  $C'$  is false in  $R_N$ .*

**Theorem 14 (Ground first-order static refutational completeness)** *Let  $\perp \notin N$  and  $N \subseteq C_{GF}$  be saturated w.r.t.  $GFInf$  and  $GFRed_1$ . Then  $R_N$  is a model of  $N$ .*

### 4.3 The Ground Higher-Order Layer

In this subsection, let  $gsel$  be a selection function on  $\mathcal{C}_{GH}$ , let  $N \subseteq \mathcal{C}_{GH}$  be a clause set saturated w.r.t.  $GHInf^{gsel}$  and  $GHRed_1^{gsel}$ , and let  $\perp \notin N$ . Clearly,  $\mathcal{F}(N)$  is then saturated w.r.t.  $GFinf^{\mathcal{F}(sel)}$  and  $GFRed_1^{\mathcal{F}(sel)}$ .

We abbreviate  $R_{\mathcal{F}(N)}$  as  $R$ . Given two terms  $s, t \in \mathcal{T}_{GH}$ , we write  $s \sim t$  to abbreviate  $R \models \mathcal{F}(s) \approx \mathcal{F}(t)$ , which is equivalent to  $\llbracket \mathcal{F}(s) \rrbracket_R = \llbracket \mathcal{F}(t) \rrbracket_R$ .

**Lemma 15** *For all terms  $t, s : \tau \rightarrow \nu$  in  $\mathcal{T}_{GH}$ , the following statements are equivalent:*

1.  $t \sim s$ ;
2.  $t(\text{diff } t s) \sim s(\text{diff } t s)$ ;
3.  $tu \sim su$  for all  $u \in \mathcal{T}_{GH}$ .

*Proof* (3)  $\Rightarrow$  (2): Take  $u := \text{diff } t s$ .

(2)  $\Rightarrow$  (1): Since  $N$  is saturated, the GEXT inference that generates the clause  $C = t(\text{diff } t s) \not\approx s(\text{diff } t s) \vee t \approx s$  is redundant—i.e.,  $C \in N \cup GHRed_{\mathcal{C}}(N)$ —and hence  $R \models \mathcal{F}(C)$  by Theorem 14. Therefore, it follows from  $t(\text{diff } t s) \sim s(\text{diff } t s)$  that  $t \sim s$ .

(1)  $\Rightarrow$  (3): We assume that  $t \sim s$ —i.e.,  $\mathcal{F}(t) \leftrightarrow_R^* \mathcal{F}(s)$ . By induction on the number of rewrite steps between  $\mathcal{F}(t)$  and  $\mathcal{F}(s)$  and by transitivity of  $\sim$ , it suffices to show that  $\mathcal{F}(t) \rightarrow_R \mathcal{F}(s)$  implies  $tu \sim su$ . If the rewrite step  $\mathcal{F}(t) \rightarrow_R \mathcal{F}(s)$  is not at the top level, then neither  $s \downarrow_{\beta\eta}$  nor  $t \downarrow_{\beta\eta}$  can be  $\lambda$ -expressions. Therefore,  $(s \downarrow_{\beta\eta})(u \downarrow_{\beta\eta})$  and  $(t \downarrow_{\beta\eta})(u \downarrow_{\beta\eta})$  are in  $\eta$ -short  $\beta$ -normal form, and there is an analogous rewrite step  $\mathcal{F}(tu) \rightarrow_R \mathcal{F}(su)$  using the same rewrite rule. It follows that  $tu \sim su$ . If the rewrite step  $\mathcal{F}(t) \rightarrow_R \mathcal{F}(s)$  is at the top level,  $\mathcal{F}(t) \rightarrow \mathcal{F}(s)$  must be a rule of  $R$ . This rule must come from a productive clause of the form  $\mathcal{F}(C) = \mathcal{F}(C' \vee t \approx s)$ . By Lemma 13,  $\mathcal{F}(t \approx s)$  is strictly eligible in  $\mathcal{F}(C)$  w.r.t.  $\mathcal{F}(gsel)$ , and hence  $t \approx s$  is strictly eligible in  $C$  w.r.t.  $gsel$ . Thus, the following GARGCONG inference  $\iota$  is applicable:

$$\frac{C' \vee t \approx s}{C' \vee tu \approx su} \text{GARGCONG}$$

By saturation,  $\iota$  is redundant w.r.t.  $N$ —i.e.,  $\text{concl}(\iota) \in N \cup GHRed_{\mathcal{C}}(N)$ . By Theorem 14,  $\mathcal{F}(\text{concl}(\iota))$  is then true in  $R$ . By Lemma 13,  $\mathcal{F}(C')$  is false in  $R$ . Therefore,  $\mathcal{F}(tu \approx su)$  must be true in  $R$ .  $\square$

**Lemma 16** *Let  $s \in \mathcal{T}_H$  and  $\theta, \theta'$  grounding substitutions such that  $x\theta \sim x\theta'$  for all variables  $x$  and  $\alpha\theta = \alpha\theta'$  for all type variables  $\alpha$ . Then  $s\theta \sim s\theta'$ .*

*Proof* In this proof, we work directly on  $\lambda$ -terms. To prove the lemma statement, it suffices to prove it for any  $\lambda$ -term  $s$ . Here, for  $\lambda$ -terms  $t_1$  and  $t_2$ , the notation  $t_1 \sim t_2$  is to be read as  $t_1 \downarrow_{\beta\eta} \sim t_2 \downarrow_{\beta\eta}$  because  $\mathcal{F}$  is only defined on  $\eta$ -short  $\beta$ -normal terms.

**DEFINITION** We extend the syntax of  $\lambda$ -terms with a new polymorphic function symbol  $\oplus : \Pi\alpha. \alpha \rightarrow \alpha \rightarrow \alpha$ . We will omit its type argument. It is equipped with two reduction rules:  $\oplus t s \rightarrow t$  and  $\oplus t s \rightarrow s$ .

The computability path order  $\succ_{\text{CPO}}$  [24] guarantees that

- $\oplus t s \succ_{\text{CPO}} s$  by applying rule  $@\triangleright$ ;
- $\oplus t s \succ_{\text{CPO}} t$  by applying rule  $@\triangleright$  twice;
- $(\lambda x. t) s \succ_{\text{CPO}} t[x \mapsto s]$  by applying rule  $@\beta$ .



Thus this order is compatible with the  $\oplus$ -reduction rules and  $\beta$ -reduction. Since this order is moreover monotone and well-founded, these reduction rules terminate. And since the reduction rules describe a finitely branching term rewrite system, by König's lemma [45], there is a maximal number of  $\beta\oplus$ -reduction steps from each  $\lambda$ -term.

DEFINITION A  $\lambda$ -term is *term-ground* if it does not contain free term variables. It may contain polymorphic type arguments.

DEFINITION We introduce an auxiliary function  $\mathcal{S}$  that essentially measures the size of a  $\lambda$ -term but assigns a size of 1 to term-ground  $\lambda$ -terms.

$$\mathcal{S}(s) = \begin{cases} 1 & \text{if } s \text{ is term-ground or is a bound or free variable or a symbol} \\ 1 + \mathcal{S}(t) & \text{if } s \text{ is not term-ground and has the form } \lambda x. t \\ \mathcal{S}(t) + \mathcal{S}(u) & \text{if } s \text{ is not term-ground and has the form } t u \end{cases}$$

We prove  $s\theta \sim s\theta'$  by well-founded induction on  $s$ ,  $\theta$ , and  $\theta'$  using the left-to-right lexicographic order on the triple  $(n_1(s), n_2(s), n_3(s)) \in \mathbb{N}^3$ , where

- $n_1(s)$  is the maximal number of  $\beta\oplus$ -reduction steps starting from  $s\sigma$ , where  $\sigma$  is the substitution mapping each term variable  $x$  to  $\oplus x\theta.x\theta'$ ;
- $n_2(s)$  is the number of term variables occurring more than once in  $s$ ;
- $n_3(s) = \mathcal{S}(s)$ .

CASE 1: The  $\lambda$ -term  $s$  is term-ground. Then the lemma is trivial.

CASE 2: The  $\lambda$ -term  $s$  contains  $k \geq 2$  free term variables. Then we can apply the induction hypothesis twice and use the transitivity of  $\sim$  as follows. Let  $x$  be one of the free term variables in  $s$ . Let  $\rho = \{x \mapsto x\theta\}$  the substitution that maps  $x$  to  $x\theta$  and ignores all other variables. Let  $\rho' = \theta'[x \mapsto x]$ .

We want to invoke the induction hypothesis on  $s\rho$  and  $s\rho'$ . This is justified because  $s\sigma \oplus$ -reduces to  $s\rho\sigma$  and to  $s\rho'\sigma$ . Hence,  $n_1(s) > n_1(s\rho)$  and  $n_1(s) > n_1(s\rho')$ .

This application of the induction hypothesis gives us  $s\rho\theta \sim s\rho\theta'$  and  $s\rho'\theta \sim s\rho'\theta'$ . Since  $s\theta = s\rho\theta$ ,  $s\rho\theta' = s\rho'\theta$ , and  $s\rho'\theta' = s\theta'$ , this implies  $s\theta \sim s\theta'$  by transitivity of  $\sim$ , as illustrated below:

$$\begin{array}{ccc} & s\rho & \\ \theta \swarrow & & \searrow \theta' \\ s\theta & \underset{\text{IH}}{\sim} & s\rho\theta' \end{array} = \begin{array}{ccc} & s\rho' & \\ \theta \swarrow & & \searrow \theta' \\ s\rho'\theta & \underset{\text{IH}}{\sim} & s\theta' \end{array}$$

CASE 3: The  $\lambda$ -term  $s$  contains a free term variable that occurs more than once. Then we rename variable occurrences apart by replacing each occurrence of each term variable  $x$  by a fresh variable  $x_i$ , for which we define  $x_i\theta = x\theta$  and  $x_i\theta' = x\theta'$ . Let  $s'$  be the resulting  $\lambda$ -term. Since  $s\sigma = s'\sigma$ , we have  $n_1(s) = n_1(s')$ . All term variables occur only once in  $s'$ . Hence,  $n_2(s) > 0 = n_2(s')$ . Therefore, we can invoke the induction hypothesis on  $s'$  to obtain  $s'\theta \sim s'\theta'$ . Since  $s\theta = s'\theta$  and  $s\theta' = s'\theta'$ , it follows that  $s\theta \sim s\theta'$ .

CASE 4: The  $\lambda$ -term  $s$  contains only one free term variable  $x$ , which occurs exactly once.

CASE 4.1: The  $\lambda$ -term  $s$  is of the form  $f(\bar{\tau})\bar{i}$  for some symbol  $f$ , some types  $\bar{\tau}$ , and some  $\lambda$ -terms  $\bar{i}$ . Then let  $u$  be the  $\lambda$ -term in  $\bar{i}$  that contains  $x$ . We want to apply the induction hypothesis to  $u$ , which can be justified as follows. Consider the longest sequence of  $\beta\oplus$ -reductions from  $u\sigma$ . This sequence can be replicated inside  $s\sigma = (f(\bar{\tau})\bar{i})\sigma$ . Therefore, the

longest sequence of  $\beta\oplus$ -reductions from  $s\sigma$  is at least as long—i.e.,  $n_1(s) \geq n_1(u)$ . Since both  $s$  and  $u$  have only one term variable occurrence, we have  $n_2(s) = 0 = n_2(u)$ . But  $n_3(s) > n_3(u)$  because  $u$  is a term-nonground subterm of  $s$ .

Applying the induction hypothesis gives us  $u\theta \sim u\theta'$ . By definition of  $\mathcal{F}$ , we have  $\mathcal{F}((f(\bar{\tau})\bar{t})\theta) = \bar{f}_m^{\bar{\tau}\theta} \mathcal{F}(\bar{t}\theta)$  and analogously for  $\theta'$ , where  $m$  is the length of  $\bar{t}$ . By congruence of  $\approx$  in first-order logic, it follows that  $s\theta \sim s\theta'$ .

CASE 4.2: The  $\lambda$ -term  $s$  is of the form  $x\bar{t}$  for some  $\lambda$ -terms  $\bar{t}$ . Then we observe that, by assumption,  $x\theta \sim x\theta'$ . By applying Lemma 15 repeatedly, we have  $x\theta\bar{t} \sim x\theta'\bar{t}$ . Since  $x$  occurs only once,  $\bar{t}$  is term-ground and hence  $s\theta = x\theta\bar{t}$  and  $s\theta' = x\theta'\bar{t}$ . Therefore,  $s\theta \sim s\theta'$ .

CASE 4.3: The  $\lambda$ -term  $s$  is of the form  $\lambda z.u$  for some  $\lambda$ -term  $u$ . Then we observe that to prove  $s\theta \sim s\theta'$ , it suffices to show that  $s\theta(\text{diff } s\theta s\theta') \sim s\theta'(\text{diff } s\theta s\theta')$  by Lemma 15. Via  $\beta\eta$ -conversion, this is equivalent to  $u\rho\theta \sim u\rho\theta'$  where  $\rho = \{z \mapsto \text{diff}(s\theta \downarrow_{\beta\eta})(s\theta' \downarrow_{\beta\eta})\}$ . To prove  $u\rho\theta \sim u\rho\theta'$ , we apply the induction hypothesis on  $u\rho$ .

It remains to show that the induction hypothesis is applicable on  $u\rho$ . Consider the longest sequence of  $\beta\oplus$ -reductions from  $u\rho\sigma$ . Since  $z\rho$  starts with the  $\text{diff}$  symbol,  $z\rho$  will not cause more  $\beta\oplus$ -reductions than  $z$ . Hence, the same sequence of  $\beta\oplus$ -reductions can be applied inside  $s\sigma = (\lambda z.u)\sigma$ , proving that  $n_1(s) \geq n_1(u\rho)$ . Since both  $s$  and  $u\rho$  have only one term variable occurrence,  $n_2(s) = 0 = n_2(u\rho)$ . But  $n_3(s) = \mathcal{S}(s) = 1 + \mathcal{S}(u)$  because  $s$  is term-nonground. Moreover,  $\mathcal{S}(u) \geq \mathcal{S}(u\rho) = n_3(u\rho)$  because  $\rho$  replaces a variable by a ground  $\lambda$ -term. Hence,  $n_3(s) > n_3(u\rho)$ , which justifies the application of the induction hypothesis.

CASE 4.4: The  $\lambda$ -term  $s$  is of the form  $(\lambda z.u)t_0\bar{t}$  for some  $\lambda$ -terms  $u$ ,  $t_0$ , and  $\bar{t}$ . We apply the induction hypothesis on  $s' = u\{z \mapsto t_0\}\bar{t}$ . To justify it, consider the longest sequence of  $\beta\oplus$ -reductions from  $s'\sigma$ . Prepending the reduction  $s\sigma \rightarrow_{\beta} s'\sigma$  to it gives us a longer sequence from  $s\sigma$ . Hence,  $n_1(s) > n_1(s')$ . The induction hypothesis gives us  $s'\theta \sim s'\theta'$ . Since  $\sim$  is invariant under  $\beta$ -reductions, it follows that  $s\theta \sim s\theta'$ .  $\square$

We proceed by defining a higher-order interpretation  $\mathcal{J}^{\text{GH}} = (\mathcal{U}^{\text{GH}}, \mathcal{J}_{\text{ty}}^{\text{GH}}, \mathcal{J}^{\text{GH}}, \mathcal{L}^{\text{GH}})$  derived from  $R$ . The interpretation  $R$  is an interpretation in monomorphic first-order logic. Let  $\mathcal{U}_{\tau}$  be its universe for type  $\tau$  and  $\mathcal{J}$  its interpretation function. When defining the universe  $\mathcal{U}^{\text{GH}}$  of the higher-order interpretation, we need to ensure that it contains subsets of function spaces, since  $\mathcal{J}_{\text{ty}}^{\text{GH}}(\rightarrow)(\mathcal{D}_1, \mathcal{D}_2)$  must be a subset of the function space from  $\mathcal{D}_1$  to  $\mathcal{D}_2$  for all  $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{U}^{\text{GH}}$ . But the first-order universes  $\mathcal{U}_{\tau}$  consist of equivalence classes of terms from  $\mathcal{T}_{\text{GF}}$  w.r.t. the rewrite system  $R$ , not of functions.

Towards this end, we will define a family of functions  $\mathcal{E}_{\tau}$  that give a meaning to the elements of the first-order universes  $\mathcal{U}_{\tau}$ . We will define a domain  $\mathcal{D}_{\tau}$  for each ground type  $\tau$  and then let  $\mathcal{U}^{\text{GH}}$  be the set of all these domains  $\mathcal{D}_{\tau}$ . Thus, there will be a one-to-one correspondence between ground types and domains.

We define  $\mathcal{E}_{\tau}$  and  $\mathcal{D}_{\tau}$  in a mutual induction and prove that  $\mathcal{E}_{\tau}$  is a bijection simultaneously. We start with nonfunctional types  $\tau$ : Let  $\mathcal{D}_{\tau} = \mathcal{U}_{\tau}$  and let  $\mathcal{E}_{\tau} : \mathcal{U}_{\tau} \rightarrow \mathcal{D}_{\tau}$  be the identity. We proceed by defining  $\mathcal{E}_{\tau \rightarrow \nu}$  and  $\mathcal{D}_{\tau \rightarrow \nu}$ . We assume that  $\mathcal{E}_{\tau}$ ,  $\mathcal{E}_{\nu}$ ,  $\mathcal{D}_{\tau}$ , and  $\mathcal{D}_{\nu}$  have already been defined and that  $\mathcal{E}_{\tau}$ ,  $\mathcal{E}_{\nu}$  are bijections. To ensure that  $\mathcal{E}_{\tau \rightarrow \nu}$  will be bijective, we first define an injective function  $\mathcal{E}_{\tau \rightarrow \nu}^0 : \mathcal{U}_{\tau \rightarrow \nu} \rightarrow \mathcal{D}_{\tau} \rightarrow \mathcal{D}_{\nu}$ , define  $\mathcal{D}_{\tau \rightarrow \nu}$  as its image  $\mathcal{E}_{\tau \rightarrow \nu}^0(\mathcal{U}_{\tau \rightarrow \nu})$ , and finally define  $\mathcal{E}_{\tau \rightarrow \nu}$  as  $\mathcal{E}_{\tau \rightarrow \nu}^0$  with its codomain restricted to  $\mathcal{D}_{\tau \rightarrow \nu}$ :

$$\begin{aligned} \mathcal{E}_{\tau \rightarrow \nu}^0 : \mathcal{U}_{\tau \rightarrow \nu} &\rightarrow \mathcal{D}_{\tau} \rightarrow \mathcal{D}_{\nu} \\ \mathcal{E}_{\tau \rightarrow \nu}^0(\llbracket \mathcal{F}(s) \rrbracket_R)(\mathcal{E}_{\tau}(\llbracket \mathcal{F}(u) \rrbracket_R)) &= \mathcal{E}_{\nu}(\llbracket \mathcal{F}(su) \rrbracket_R) \end{aligned}$$

This is a valid definition because each element of  $\mathcal{U}_{\tau \rightarrow \nu}$  is of the form  $\llbracket \mathcal{F}(s) \rrbracket_R$  for some  $s$  and each element of  $\mathcal{D}_\tau$  is of the form  $\mathcal{E}_\tau(\llbracket \mathcal{F}(u) \rrbracket_R)$  for some  $u$ . This function is well defined if it does not depend on the choice of  $s$  and  $u$ . To show this, we assume that there are other ground terms  $t$  and  $v$  such that  $\llbracket \mathcal{F}(s) \rrbracket_R = \llbracket \mathcal{F}(t) \rrbracket_R$  and  $\mathcal{E}_\tau(\llbracket \mathcal{F}(u) \rrbracket_R) = \mathcal{E}_\tau(\llbracket \mathcal{F}(v) \rrbracket_R)$ . Since  $\mathcal{E}_\tau$  is bijective, we have  $\llbracket \mathcal{F}(u) \rrbracket_R = \llbracket \mathcal{F}(v) \rrbracket_R$ . Applying Lemma 16 to the term  $xy$  and the substitutions  $\{x \mapsto s, y \mapsto u\}$  and  $\{x \mapsto t, y \mapsto v\}$ , we obtain  $\llbracket \mathcal{F}(su) \rrbracket_R = \llbracket \mathcal{F}(tv) \rrbracket_R$ , indicating that  $\mathcal{E}_{\tau \rightarrow \nu}^0$  is well defined. It remains to show that  $\mathcal{E}_{\tau \rightarrow \nu}^0$  is injective as a function from  $\mathcal{U}_{\tau \rightarrow \nu}$  to  $\mathcal{D}_\tau \rightarrow \mathcal{D}_\nu$ . Assume two terms  $s, t \in \mathcal{T}_{\text{GH}}$  such that for all  $u \in \mathcal{T}_{\text{GH}}$ , we have  $\llbracket \mathcal{F}(su) \rrbracket_R = \llbracket \mathcal{F}(tu) \rrbracket_R$ . By Lemma 15, it follows that  $\llbracket \mathcal{F}(s) \rrbracket_R = \llbracket \mathcal{F}(t) \rrbracket_R$ , which concludes the proof that  $\mathcal{E}_{\tau \rightarrow \nu}^0$  is injective.

We define  $\mathcal{D}_{\tau \rightarrow \nu} = \mathcal{E}_{\tau \rightarrow \nu}^0(\mathcal{U}_{\tau \rightarrow \nu})$  and  $\mathcal{E}_{\tau \rightarrow \nu}(a) = \mathcal{E}_{\tau \rightarrow \nu}^0(a)$ . This ensures that  $\mathcal{E}_{\tau \rightarrow \nu}$  is bijective and concludes the inductive definition of  $\mathcal{D}$  and  $\mathcal{E}$ . In the following, we will usually write  $\mathcal{E}$  instead of  $\mathcal{E}_\tau$ , since the type  $\tau$  is determined by first argument of  $\mathcal{E}_\tau$ .

We define the higher-order universe as  $\mathcal{U}^{\text{GH}} = \{\mathcal{D}_\tau \mid \tau \text{ ground}\}$ . Moreover, we define  $\mathcal{J}_{\text{ty}}^{\text{GH}}(\kappa)(\mathcal{D}_{\bar{\tau}}) = \mathcal{U}_{\kappa(\bar{\tau})}$  for all  $\kappa \in \Sigma_{\text{ty}}$ , completing the type interpretation  $\mathcal{J}_{\text{ty}}^{\text{GH}} = (\mathcal{U}^{\text{GH}}, \mathcal{J}_{\text{ty}}^{\text{GH}})$ .

We define the interpretation function as  $\mathcal{J}^{\text{GH}}(\mathfrak{f}, \mathcal{D}_{\bar{v}_m}) = \mathcal{E}(\mathcal{J}(\mathfrak{f}_0^{\bar{v}_m}))$  for all  $\mathfrak{f} : \Pi \bar{\alpha}_m. \tau$ .

Finally, we need to define the designation function  $\mathcal{L}^{\text{GH}}$ , which takes a valuation  $\xi$  and a  $\lambda$ -expression as arguments. Given a valuation  $\xi$ , we choose a grounding substitution  $\theta$  such that  $\mathcal{D}_{\alpha\theta} = \xi(\alpha)$  and  $\mathcal{E}(\llbracket \mathcal{F}(x\theta) \rrbracket_R) = \xi(x)$  for all type variables  $\alpha$  and all variables  $x$ . Such a substitution can be constructed as follows: We can fulfill the first equation in a unique way because there is a one-to-one correspondence between ground types and domains. Since  $\mathcal{E}^{-1}(\xi(x))$  is an element of a first-order universe and  $R$  is term-generated, there exists a ground term  $t$  such that  $\llbracket t \rrbracket_R^\xi = \mathcal{E}^{-1}(\xi(x))$ . Choosing one such  $t$  and defining  $x\theta = \mathcal{F}^{-1}(t)$  gives us a grounding substitution  $\theta$  with the desired property.

We define  $\mathcal{L}^{\text{GH}}(\xi, (\lambda x. t)) = \mathcal{E}(\llbracket \mathcal{F}((\lambda x. t)\theta) \rrbracket_R)$ . To prove that this is well defined, we assume that there exists another substitution  $\theta'$  with the properties  $\mathcal{D}_{\alpha\theta'} = \xi(\alpha)$  for all  $\alpha$  and  $\mathcal{E}(\llbracket \mathcal{F}(x\theta') \rrbracket_R) = \xi(x)$  for all  $x$ . Then we have  $\alpha\theta = \alpha\theta'$  for all  $\alpha$  due to the one-to-one correspondence between domains and ground types. We have  $\llbracket \mathcal{F}(x\theta) \rrbracket_R = \llbracket \mathcal{F}(x\theta') \rrbracket_R$  for all  $x$  because  $\mathcal{E}$  is injective. By Lemma 16 it follows that  $\llbracket \mathcal{F}((\lambda x. t)\theta) \rrbracket_R = \llbracket \mathcal{F}((\lambda x. t)\theta') \rrbracket_R$ , which proves that  $\mathcal{L}^{\text{GH}}$  is well defined.

This concludes the definition of the interpretation  $\mathcal{J}^{\text{GH}} = (\mathcal{U}^{\text{GH}}, \mathcal{J}_{\text{ty}}^{\text{GH}}, \mathcal{J}^{\text{GH}}, \mathcal{L}^{\text{GH}})$ . It remains to show that  $\mathcal{J}^{\text{GH}}$  is proper. In a proper interpretation, the denotation  $\llbracket t \rrbracket_{\mathcal{J}^{\text{GH}}}$  of a term  $t$  does not depend on the representative of  $t$  modulo  $\beta\eta$ , but since we have not yet shown  $\mathcal{J}^{\text{GH}}$  to be proper, we cannot rely on this property. For this reason, we use  $\lambda$ -terms in the following three lemmas and mark all  $\beta\eta$ -reductions explicitly.

The higher-order interpretation  $\mathcal{J}^{\text{GH}}$  relates to the first-order interpretation  $R$  as follows:

**Lemma 17** *Given a ground  $\lambda$ -term  $t$ , we have*

$$\llbracket t \rrbracket_{\mathcal{J}^{\text{GH}}} = \mathcal{E}(\llbracket \mathcal{F}(t \downarrow_{\beta\eta}) \rrbracket_R)$$

*Proof* By induction on  $t$ . Assume that  $\llbracket s \rrbracket_{\mathcal{J}^{\text{GH}}} = \mathcal{E}(\llbracket \mathcal{F}(s \downarrow_{\beta\eta}) \rrbracket_R)$  for all proper subterms  $s$  of  $t$ . If  $t$  is of the form  $\mathfrak{f}(\bar{\tau})$ , then

$$\begin{aligned} \llbracket t \rrbracket_{\mathcal{J}^{\text{GH}}} &= \mathcal{J}^{\text{GH}}(\mathfrak{f}, \mathcal{D}_{\bar{\tau}}) \\ &= \mathcal{E}(\mathcal{J}(\mathfrak{f}_0, \mathcal{U}_{\mathcal{F}(\bar{\tau})})) \\ &= \mathcal{E}(\llbracket \mathfrak{f}_0 \langle \mathcal{F}(\bar{\tau}) \rangle \rrbracket_R) \\ &= \mathcal{E}(\llbracket \mathcal{F}(\mathfrak{f}(\bar{\tau})) \rrbracket_R) \\ &= \mathcal{E}(\llbracket \mathcal{F}(\mathfrak{f}(\bar{\tau}) \downarrow_{\beta\eta}) \rrbracket_R) = \mathcal{E}(\llbracket \mathcal{F}(t \downarrow_{\beta\eta}) \rrbracket_R) \end{aligned}$$

If  $t$  is an application  $t = t_1 t_2$ , where  $t_1$  is of type  $\tau \rightarrow \nu$ , then

$$\begin{aligned} \llbracket t_1 t_2 \rrbracket_{\mathcal{J}^{\text{GH}}} &= \llbracket t_1 \rrbracket_{\mathcal{J}^{\text{GH}}} (\llbracket t_2 \rrbracket_{\mathcal{J}^{\text{GH}}}) \\ &\stackrel{\text{IH}}{=} \mathcal{E}_{\tau \rightarrow \nu} (\llbracket \mathcal{F}(t_1 \downarrow_{\beta\eta}) \rrbracket_R) (\mathcal{E}_{\tau} (\llbracket \mathcal{F}(t_2 \downarrow_{\beta\eta}) \rrbracket_R)) \\ &\stackrel{\text{Def } \mathcal{E}}{=} \mathcal{E}_{\nu} (\llbracket \mathcal{F}(t_1 t_2 \downarrow_{\beta\eta}) \rrbracket_R) \end{aligned}$$

If  $t$  is a  $\lambda$ -expression, then

$$\begin{aligned} \llbracket \lambda x. u \rrbracket_{\mathcal{J}^{\text{GH}}}^{\xi} &= \mathcal{L}^{\text{GH}}(\xi, (\lambda x. u)) \\ &= \mathcal{E}(\llbracket \mathcal{F}((\lambda x. u)\theta) \rrbracket_R) \\ &= \mathcal{E}(\llbracket \mathcal{F}(\lambda x. u \downarrow_{\beta\eta}) \rrbracket_R) \end{aligned}$$

where  $\theta$  is a substitution such that  $\mathcal{D}_{\alpha\theta} = \xi(\alpha)$  and  $\mathcal{E}(\llbracket \mathcal{F}(x\theta) \rrbracket_R) = \xi(x)$ .  $\square$

We need to show that the interpretation  $\mathcal{J}^{\text{GH}} = (\mathcal{U}^{\text{GH}}, \mathcal{J}_{\text{ty}}^{\text{GH}}, \mathcal{J}^{\text{GH}}, \mathcal{L}^{\text{GH}})$  is proper. In the proof, we will need to employ the following lemma, which is very similar to the substitution lemma (Lemma 7), but we must prove it here for our particular interpretation  $\mathcal{J}^{\text{GH}}$  because we have not shown that  $\mathcal{J}^{\text{GH}}$  is proper yet.

**Lemma 18 (Substitution lemma)**  $\llbracket \tau \rho \rrbracket_{\mathcal{J}_{\text{ty}}^{\xi}}^{\xi} = \llbracket \tau \rrbracket_{\mathcal{J}_{\text{ty}}^{\xi'}}^{\xi'}$  and  $\llbracket t \rho \rrbracket_{\mathcal{J}^{\xi}}^{\xi} = \llbracket t \rrbracket_{\mathcal{J}^{\xi'}}^{\xi'}$  for all  $\lambda$ -terms  $t$ , all  $\tau \in \mathcal{T}y_{\text{H}}$  and all grounding substitutions  $\rho$ , where  $\xi'(\alpha) = \llbracket \alpha \rho \rrbracket_{\mathcal{J}_{\text{ty}}^{\xi}}^{\xi}$  for all type variables  $\alpha$  and  $\xi'(x) = \llbracket x \rho \rrbracket_{\mathcal{J}^{\xi}}^{\xi}$  for all term variables  $x$ .

*Proof* We proceed by induction on the structure of  $\tau$  and  $t$ . The proof is identical to the one of Lemma 6, except for the last step, which uses properness of the interpretation, a property we cannot assume here. However, here, we have the assumption that  $\rho$  is a grounding substitution. Therefore, if  $t$  is a  $\lambda$ -expression, we argue as follows:

$$\begin{aligned} \llbracket (\lambda z. u) \rho \rrbracket_{\mathcal{J}^{\xi}}^{\xi} &= \llbracket (\lambda z. u \rho') \rrbracket_{\mathcal{J}^{\xi}}^{\xi} && \text{where } \rho'(z) = z \text{ and } \rho'(x) = \rho(x) \text{ for } x \neq z \\ &= \mathcal{L}^{\text{GH}}(\xi, (\lambda z. u \rho')) && \text{by the definition of the term denotation} \\ &= \mathcal{E}(\llbracket \mathcal{F}((\lambda z. u) \rho \theta_{\xi} \downarrow_{\beta\eta}) \rrbracket_R^{\xi}) && \text{by the definition of } \mathcal{L}^{\text{GH}} \\ &= \mathcal{E}(\llbracket \mathcal{F}((\lambda z. u) \rho \downarrow_{\beta\eta}) \rrbracket_R^{\xi}) && \text{because } (\lambda z. u) \rho \text{ is ground} \\ &\stackrel{*}{=} \mathcal{L}^{\text{GH}}(\xi', \lambda z. u) && \text{by the definition of } \mathcal{L}^{\text{GH}} \text{ and Lemma 17} \\ &= \llbracket \lambda z. u \rrbracket_{\mathcal{J}^{\xi'}}^{\xi'} && \text{by the definition of the term denotation} \end{aligned}$$

The step  $*$  is justified as follows: We have  $\mathcal{L}^{\text{GH}}(\xi', \lambda z. u) = \mathcal{E}(\llbracket \mathcal{F}((\lambda z. u)\theta) \rrbracket_R^{\xi'})$  by the definition of  $\mathcal{L}^{\text{GH}}$ , if  $\theta$  is a substitution such that  $\mathcal{D}_{\alpha\theta} = \xi'(\alpha)$  for all  $\alpha$  and  $\mathcal{E}(\llbracket \mathcal{F}(x\theta) \rrbracket_R^{\xi'}) = \xi'(x)$  for all  $x$ . By the definition of  $\xi'$  and by Lemma 17,  $\rho$  is such a substitution. Hence,  $\mathcal{L}^{\text{GH}}(\xi', \lambda z. u) = \mathcal{E}(\llbracket \mathcal{F}((\lambda z. u) \rho \downarrow_{\beta\eta}) \rrbracket_R^{\xi'})$ .  $\square$

**Lemma 19** *The interpretation  $\mathcal{J}^{\text{GH}}$  is proper.*

*Proof* We must show that  $\llbracket (\lambda x. t) \rrbracket_{\mathcal{J}^{\text{GH}}}^{\xi}(a) = \llbracket t \rrbracket_{\mathcal{J}^{\text{GH}}}^{\xi[x \mapsto a]}$  for all  $\lambda$ -expressions  $\lambda x. t$ , all substitutions  $\xi$ , and all values  $a$ .

$$\begin{aligned}
\llbracket \lambda x. t \rrbracket_{\mathcal{J}^{\text{GH}}}^{\xi}(a) &= \mathcal{L}^{\text{GH}}(\xi, \lambda x. t)(a) && \text{by the definition of } \llbracket \cdot \rrbracket_{\mathcal{J}^{\text{GH}}} \\
&= \mathcal{E}(\llbracket \mathcal{F}((\lambda x. t)\theta \downarrow_{\beta\eta}) \rrbracket_R)(a) && \text{by the definition of } \mathcal{L}^{\text{GH}} \text{ for some } \theta \\
&&& \text{such that } \mathcal{E}(\llbracket \mathcal{F}(z\theta) \rrbracket_R) = \xi(z) \text{ for all } z \\
&&& \text{and } \mathcal{D}_{\alpha\theta} = \xi(\alpha) \text{ for all } \alpha \\
&= \mathcal{E}(\llbracket \mathcal{F}((\lambda x. t)\theta s \downarrow_{\beta\eta}) \rrbracket_R) && \text{by the definition of } \mathcal{E} \\
&&& \text{where } \mathcal{E}(\llbracket \mathcal{F}(s) \rrbracket_R) = a \\
&= \mathcal{E}(\llbracket \mathcal{F}(t(\theta[x \mapsto s]) \downarrow_{\beta\eta}) \rrbracket_R) && \text{by } \beta\text{-reduction} \\
&= \llbracket t(\theta[x \mapsto s]) \rrbracket_{\mathcal{J}^{\text{GH}}} && \text{by Lemma 17} \\
&= \llbracket t \rrbracket_{\mathcal{J}^{\text{GH}}}^{\xi[x \mapsto a]} && \text{by Lemma 18} \quad \square
\end{aligned}$$

**Lemma 20**  $\mathcal{J}^{\text{GH}}$  is a model of  $N$ .

*Proof* By Lemma 17, we have  $\llbracket t \rrbracket_{\mathcal{J}^{\text{GH}}} = \mathcal{E}(\llbracket \mathcal{F}(t) \rrbracket_R)$  for all  $t \in \mathcal{T}_{\text{GH}}$ . Since  $\mathcal{E}$  is a bijection, it follows that any (dis)equation  $s \approx t \in C_{\text{GH}}$  is true in  $\mathcal{J}^{\text{GH}}$  if and only if  $\mathcal{F}(s \approx t)$  is true in  $R$ . Hence, a clause  $C \in C_{\text{GH}}$  is true in  $\mathcal{J}^{\text{GH}}$  if and only if  $\mathcal{F}(C)$  is true in  $R$ . By Theorem 14,  $R$  is a model of  $\mathcal{F}(N)$ —that is, for all clauses  $C \in N$ ,  $\mathcal{F}(C)$  is true in  $R$ . Hence, all clauses  $C \in N$  are true in  $\mathcal{J}^{\text{GH}}$  and therefore  $\mathcal{J}^{\text{GH}}$  is a model of  $N$ .  $\square$

We summarize the results of this subsection in the following theorem:

**Theorem 21 (Ground static refutational completeness)** *Let  $N \subseteq C_{\text{GH}}$  be a clause set saturated w.r.t.  $\text{GHInf}^{\text{gsel}}$  and  $\text{GHRed}_1^{\text{gsel}}$  for some selection function  $\text{gsel}$ . Then  $N \models \perp$  if and only if  $\perp \in N$ .*

#### 4.4 The Nonground Higher-Order Layer

To lift the result to the nonground layer, we employ the saturation framework of Waldmann et al. [67]. It is easy to see that the entailment relation  $\models$  on GH is a consequence relation in the sense of the framework. We need to show that our redundancy criterion on GH is a redundancy criterion in the sense of the framework and that  $\mathcal{G}$  is a grounding function in the sense of the framework:

**Lemma 22** *Given a clausal higher-order interpretation  $\mathcal{J}$  on GH, there exists a first-order interpretation  $\mathcal{J}^{\text{GF}}$  on GF such that for any clause  $C \in C_{\text{GH}}$  the truth values of  $C$  in  $\mathcal{J}$  and of  $\mathcal{F}(C)$  in  $\mathcal{J}^{\text{GF}}$  coincide.*

*Proof* Let  $\mathcal{J} = (\mathcal{J}_{\text{ty}}, \mathcal{J}, \mathcal{L})$  be a clausal higher-order interpretation on GH. Let  $\mathcal{U}_{\tau}^{\text{GF}} = \llbracket \tau \rrbracket_{\mathcal{J}_{\text{ty}}}$  be the first-order type universe for the ground type  $\tau$ . For a symbol  $f_j^{\bar{v}} \in \Sigma_{\text{GF}}$ , let  $\mathcal{J}^{\text{GF}}(f_j^{\bar{v}}) = \llbracket f(\bar{v}) \rrbracket_{\mathcal{J}}$  (up to currying). For a symbol  $\text{lam}_{\lambda x. t} \in \Sigma_{\text{GF}}$ , let  $\mathcal{J}^{\text{GF}}(\text{lam}_{\lambda x. t}) = \llbracket \lambda x. t \rrbracket_{\mathcal{J}}$ . This defines a first-order interpretation  $\mathcal{J}^{\text{GF}} = (\mathcal{U}^{\text{GF}}, \mathcal{J}^{\text{GF}})$ .

We need to show that for any  $C \in C_{\text{GH}}$ ,  $\mathcal{J} \models C$  if and only if  $\mathcal{J}^{\text{GF}} \models \mathcal{F}(C)$ . It suffices to show that  $\llbracket t \rrbracket_{\mathcal{J}} = \llbracket \mathcal{F}(t) \rrbracket_{\mathcal{J}^{\text{GF}}}$  for all terms  $t \in \mathcal{T}_{\text{GH}}$ . We prove this by induction on the structure of the  $\eta$ -short  $\beta$ -normal form of  $t$ . If  $t$  is a  $\lambda$ -expression, this is obvious. If  $t$  is of the form  $f(\bar{v})\bar{s}_j$ , then  $\mathcal{F}(t) = f_j^{\bar{v}}(\mathcal{F}(\bar{s}_j))$  and hence  $\llbracket \mathcal{F}(t) \rrbracket_{\mathcal{J}^{\text{GF}}} = \mathcal{J}^{\text{GF}}(f_j^{\bar{v}})(\llbracket \mathcal{F}(\bar{s}_j) \rrbracket_{\mathcal{J}^{\text{GF}}}) = \llbracket f(\bar{v}) \rrbracket_{\mathcal{J}}(\llbracket \mathcal{F}(\bar{s}_j) \rrbracket_{\mathcal{J}^{\text{GF}}}) \stackrel{\text{IH}}{=} \llbracket f(\bar{v}) \rrbracket_{\mathcal{J}}(\llbracket \bar{s}_j \rrbracket_{\mathcal{J}}) = \llbracket t \rrbracket_{\mathcal{J}}$ .  $\square$

**Lemma 23** *The redundancy criterion for GH is a redundancy criterion in the sense of the saturation framework.*

*Proof* Analogous to the proof of Lemma 4.10 of Bentkamp et al. [12].  $\square$

**Lemma 24** *Given a selection function  $sel$  on  $C_H$ , the grounding functions  $\mathcal{G}^{gsel}$  for  $gsel \in \mathcal{G}(sel)$  are grounding functions in the sense of the saturation framework.*

*Proof* Clearly,  $C = \perp$  if and only if  $\mathcal{G}(C) = \perp$ , proving (G1) and (G2). For every  $\iota \in HInf^{sel}$ , by the definition of  $\mathcal{G}^{gsel}$ , we have  $concl(\mathcal{G}^{gsel}(\iota)) \subseteq \mathcal{G}(concl(\iota))$ , and thus (G3').  $\square$

Let  $sel$  be a selection function on  $C_H$  fulfilling the selection restriction that a literal  $L\langle y \rangle$  may not be selected if  $y\bar{u}_n$ , with  $n > 0$ , is a  $\succsim$ -maximal term of the clause. Let  $N \subseteq C_H$  be a clause set saturated w.r.t.  $HInf^{sel}$  and  $HRed_1^{sel}$ . For the lifting mechanism of the saturation framework to apply, we need to show that there exists a selection function  $gsel \in \mathcal{G}(sel)$  such that all inferences  $\iota \in GHInf^{gsel}$  with  $prems(\iota) \in \mathcal{G}(N)$  are liftable or redundant. Here, for  $\iota$  to be *liftable* means that  $\iota$  is a  $\mathcal{G}^{gsel}$ -ground instance of a  $HInf^{sel}$ -inference from  $N$ ; for  $\iota$  to be *redundant* means that  $\iota \in GHRed_1^{gsel}(\mathcal{G}(N))$ .

To choose the right selection function  $gsel \in \mathcal{G}(sel)$ , we observe that each ground clause  $C \in \mathcal{G}(N)$  must have at least one corresponding clause  $D \in N$  such that  $C$  is a ground instance of  $D$ . We choose one of them for each  $C \in \mathcal{G}(N)$ , which we denote  $\mathcal{G}^{-1}(C)$ . Then let  $gsel$  select those literals in  $C$  that correspond to the literals selected by  $sel$  in  $\mathcal{G}^{-1}(C)$ . With respect to this selection function  $gsel$ , we can show that all inferences from  $\mathcal{G}(N)$  are liftable or redundant:

**Lemma 25 (Lifting of EQRES, EQFACT, GARGCONG, and GEXT)** *All EQRES, EQFACT, GARGCONG, and GEXT inferences are liftable.*

*Proof* EQRES: Let  $\iota \in GHInf^{gsel}$  be an EQRES inference with  $prems(\iota) \in \mathcal{G}(N)$ . Then  $\iota$  is of the form

$$\frac{C\theta = C'\theta \vee s\theta \not\approx s'\theta}{C'\theta} \text{EQRES}$$

where  $\mathcal{G}^{-1}(C\theta) = C = C' \vee s \not\approx s'$  and the literal  $s\theta \not\approx s'\theta$  is eligible w.r.t.  $gsel$ . Since  $s\theta$  and  $s'\theta$  are unifiable and ground, we have  $s\theta = s'\theta$ . Thus, there exists some corresponding unifier  $\sigma \in CSU(s, s')$ . Then  $s \not\approx s'$  is eligible in  $C$  w.r.t.  $\sigma$  and  $sel$ . Hence, the following inference  $\iota' \in HInf^{sel}$  is applicable:

$$\frac{C' \vee s \not\approx s'}{C'\sigma} \text{EQRES}$$

Since  $\sigma \in CSU(s, s')$ , we have  $x\sigma\sigma = x\sigma$  and  $x\sigma\rho = x\theta$  for all variables  $x$  in  $C$  for some substitution  $\rho$ . Therefore,  $\iota$  is the  $\sigma\rho$ -ground instance of  $\iota'$  and is therefore liftable.

EQFACT: Analogously, if  $\iota \in GHInf^{gsel}$  is an EQFACT inference with  $prems(\iota) \in \mathcal{G}(N)$ , then  $\iota$  is of the form

$$\frac{C\theta = C'\theta \vee s'\theta \approx t'\theta \vee s\theta \approx t\theta}{C'\theta \vee t\theta \not\approx t'\theta \vee s\theta \approx t'\theta} \text{EQFACT}$$

where  $\mathcal{G}^{-1}(C\theta) = C = C' \vee s' \approx t' \vee s \approx t$ , the literal  $s\theta \approx t\theta$  is eligible in  $C$  w.r.t.  $gsel$ , and  $s\theta \not\approx t\theta$ . Then  $s \not\approx t$ . Moreover,  $s\theta$  and  $s'\theta$  are unifiable and ground. Hence,  $s\theta = s'\theta$  and

there exists some corresponding  $\sigma \in \text{CSU}(s, s')$ . Then  $s \approx t$  is eligible in  $C$  w.r.t.  $\sigma$  and  $\text{sel}$ . It follows that the following inference  $\iota' \in \text{HInf}^{\text{sel}}$  is applicable:

$$\frac{C' \vee s' \approx t' \vee s \approx t}{(C' \vee t \not\approx t' \vee s \approx t')\sigma} \text{EQFACT}$$

Since  $\sigma \in \text{CSU}(s, s')$ , we have  $x\sigma\sigma = x\sigma$  and  $x\sigma\rho = x\theta$  for all variables  $x$  in  $C$  for some substitution  $\rho$ . Hence,  $\iota$  is the  $\sigma\rho$ -ground instance of  $\iota'$  and is therefore liftable.

**GARGCONG:** Let  $\iota \in \text{GHInf}^{\text{gsel}}$  be a GARGCONG inference with  $\text{prems}(\iota) \in \mathcal{G}(N)$ . Then  $\iota$  is of the form

$$\frac{C\theta = C'\theta \vee s\theta \approx s'\theta}{C'\theta \vee s\theta \bar{u}_n \approx s'\theta \bar{u}_n} \text{GARGCONG}$$

where  $\mathcal{G}^{-1}(C\theta) = C = C' \vee s \approx s'$ , the literal  $s\theta \approx s'\theta$  is strictly eligible w.r.t.  $\text{gsel}$ , and  $s\theta$  and  $s'\theta$  are of functional type. It follows that  $s$  and  $s'$  have either a functional or a polymorphic type. Let  $\sigma$  be the most general substitution such that  $s\sigma$  and  $s'\sigma$  take  $n$  arguments. Then  $s \not\approx s'$  is eligible in  $C$  w.r.t.  $\sigma$  and  $\text{sel}$ . Hence the following inference  $\iota' \in \text{HInf}^{\text{sel}}$  is applicable:

$$\frac{C' \vee s \approx s'}{(C' \vee s \bar{x}_n \approx s' \bar{x}_n)\sigma} \text{ARGCONG}$$

Since  $\sigma$  is the most general substitution that ensures well-typedness of the conclusion,  $\iota$  is a ground instance of  $\iota'$  and is therefore liftable.

**GEXT:** The conclusion of a GEXT inference in  $\text{GHInf}$  is by definition a ground instance of the conclusion of a EXT inference in  $\text{HInf}$ . Hence, the GEXT inference is a ground instance of the EXT inference. Therefore it is liftable.  $\square$

Some of the SUP inferences in  $\text{GHInf}$  are liftable as well:

**Lemma 26 (Instances of green subterms)** Let  $s$  be a  $\lambda$ -term in  $\eta$ -short  $\beta$ -normal form, let  $\sigma$  be a substitution, let  $p$  be a green position of both  $s$  and  $s\sigma \downarrow_{\beta\eta}$ . Then  $(s|_p)\sigma \downarrow_{\beta\eta} = (s\sigma \downarrow_{\beta\eta})|_p$ .

*Proof* By induction on  $p$ . If  $p = \varepsilon$ , then  $(s|_p)\sigma \downarrow_{\beta\eta} = s\sigma \downarrow_{\beta\eta} = (s\sigma \downarrow_{\beta\eta})|_p$ . If  $p = i.p'$ , then  $s = f(\bar{\tau})s_1 \dots s_n$  and  $s\sigma = f(\bar{\tau}\sigma)(s_1\sigma) \dots (s_n\sigma)$ , where  $1 \leq i \leq n$  and  $p'$  is a green position of  $s_i$ . Clearly,  $\beta\eta$ -normalization steps of  $s\sigma$  can take place only in proper subterms. So  $s\sigma \downarrow_{\beta\eta} = f(\bar{\tau}\sigma)(s_1\sigma \downarrow_{\beta\eta}) \dots (s_n\sigma \downarrow_{\beta\eta})$ . Since  $p = i.p'$  is a green position of  $s\sigma \downarrow_{\beta\eta}$ ,  $p'$  must be a green position of  $(s_i\sigma) \downarrow_{\beta\eta}$ . By induction,  $(s_i|_{p'})\sigma \downarrow_{\beta\eta} = (s_i\sigma \downarrow_{\beta\eta})|_{p'}$ . Therefore  $(s|_p)\sigma \downarrow_{\beta\eta} = (s|_{i.p'})\sigma \downarrow_{\beta\eta} = (s_i|_{p'})\sigma \downarrow_{\beta\eta} = (s_i\sigma \downarrow_{\beta\eta})|_{p'} = (s\sigma \downarrow_{\beta\eta})|_p$ .  $\square$

**Lemma 27 (Lifting of SUP)** Let  $\iota \in \text{GHInf}^{\text{gsel}}$  be a SUP inference

$$\frac{\overbrace{D'\theta \vee t\theta \approx t'\theta}^{D\theta} \quad \overbrace{C'\theta \vee s\theta \langle t\theta \rangle_p \approx s'\theta}^{C\theta}}{D'\theta \vee C'\theta \vee s\theta \langle t'\theta \rangle_p \approx s'\theta} \text{SUP}$$

where  $\mathcal{G}^{-1}(D\theta) = D = D' \vee t \approx t'$  and  $\mathcal{G}^{-1}(C\theta) = C = C' \vee s \approx s'$ . We assume that  $s$ ,  $t$ ,  $s\theta$ , and  $t\theta$  are represented by  $\lambda$ -terms in  $\eta$ -short  $\beta$ -normal form. Let  $p'$  be the longest prefix of  $p$  that is a green position of  $s$ . Since  $\varepsilon$  is a green position of  $s$ , the longest prefix always exists. Let  $u = s|_{p'}$ . Suppose one of the following conditions applies: (i)  $u$  is a deep variable in  $C$ ; (ii)  $p = p'$  and the variable condition holds for  $D$  and  $C$  or (iii)  $p \neq p'$  and  $u$  is not a variable. Then  $\iota$  is liftable.

*Proof* The SUP inference conditions for  $\iota$  are that  $t\theta \approx t'\theta$  is strictly eligible,  $s\theta \approx s'\theta$  is strictly eligible if positive and eligible if negative,  $D\theta \not\prec C\theta$ ,  $t\theta \not\prec t'\theta$ , and  $s\theta \not\prec s'\theta$ . We assume that  $s$ ,  $t$ ,  $s\theta$ , and  $t\theta$  are represented by  $\lambda$ -terms in  $\eta$ -short  $\beta$ -normal form. By Lemma 26,  $u\theta$  agrees with  $s\theta|_{p'}$  (considering both as terms rather than as  $\lambda$ -terms).

CASE 1: We have (a)  $p = p'$ , and (b)  $u$  is not fluid, and (c) if  $u$  is a variable, it is not deep in  $C$ . Then  $u\theta = s\theta|_{p'} = s\theta|_p = t\theta$ . Since  $\theta$  is a unifier of  $u$  and  $t$ , there exists some corresponding  $\sigma \in \text{CSU}(t, u)$ . The inference conditions can be lifted: (Strict) eligibility of  $t\theta \approx t'\theta$  and  $s\theta \approx s'\theta$  w.r.t.  $gsel$  implies (strict) eligibility of  $t \approx t'$  and  $s \approx s'$  w.r.t.  $\sigma$  and  $sel$ ;  $D\theta \not\prec C\theta$  implies  $D \not\prec C$ ;  $t\theta \not\prec t'\theta$  implies  $t \not\prec t'$ ; and  $s\theta \not\prec s'\theta$  implies  $s \not\prec s'$ . Moreover, by (a) and (c), condition (ii) must hold and thus the variable condition holds for  $D$  and  $C$ . Hence there is the following SUP inference  $\iota' \in HInf$ :

$$\frac{D' \vee t \approx t' \quad C' \vee s\langle u \rangle_p \approx s'}{(D' \vee C' \vee s\langle t' \rangle_p \approx s')\sigma} \text{ SUP}$$

Since  $\sigma \in \text{CSU}(t, u)$ , we have  $x\sigma\sigma = x\sigma$  and  $x\sigma\rho = x\theta$  for all variables  $x$  occurring in  $D$  and  $C$  for some substitution  $\rho$ . Hence,  $\iota$  is the  $\sigma\rho$ -ground instance of  $\iota'$  and therefore liftable.

CASE 2: We have (a)  $p \neq p'$ ; or (b)  $u$  is fluid; or (c)  $u$  is a deep variable in  $C$ . We will first show that (a) implies (b) or (c). Suppose (a) holds but neither (b) nor (c) holds. Then condition (iii) must hold—i.e.,  $u$  is not a variable. Moreover, since (b) does not hold,  $u$  cannot have the form  $y\bar{u}_n$  for a variable  $y$  and  $n \geq 1$ . If  $u$  were of the form  $f\langle \bar{\tau} \rangle s_1 \dots s_n$  with  $n \geq 0$ ,  $u\theta$  would have the form  $f\langle \bar{\tau}\theta \rangle (s_1\theta) \dots (s_n\theta)$ , but then there is some  $1 \leq i \leq n$  such that  $p'.i$  is a prefix of  $p$  and  $s|_{p'.i}$  is a green subterm of  $s$ , contradicting the maximality of  $p'$ . So  $u$  must be a  $\lambda$ -expression, but since  $t\theta$  is a proper green subterm of  $u\theta$ ,  $u\theta$  cannot be a  $\lambda$ -expression, yielding a contradiction. We may thus assume that (b) or (c) holds.

Let  $p = p'.p''$ . Let  $z$  be a fresh variable. Define a substitution  $\theta'$  that maps this variable  $z$  to  $\lambda y. (s\theta|_{p'})\langle y \rangle_{p''}$  and any other variable  $w$  to  $w\theta$ . Clearly,  $(zt)\theta' = (s\theta|_{p'})\langle t\theta \rangle_{p''} = s\theta|_{p'} = u\theta = u\theta'$ . Since  $\theta'$  is a unifier of  $u$  and  $zt$ , there exists some corresponding  $\sigma \in \text{CSU}(zt, u)$ . As in case 1, (strict) eligibility of the ground literals implies (strict) eligibility of the nonground literals. Moreover, by construction of  $\theta'$ ,  $t\theta' = t\theta \neq t'\theta = t'\theta'$  implies  $(zt)\theta' \neq (z't')\theta'$ , and thus  $(zt)\sigma \neq (z't')\sigma$ . Since we also have (b) or (c), there is the following inference  $\iota'$ :

$$\frac{D' \vee t \approx t' \quad C' \vee s\langle u \rangle_{p'} \approx s'}{(D' \vee C' \vee s\langle z't' \rangle_{p'} \approx s')\sigma} \text{ FLUIDSUP}$$

Since  $\sigma \in \text{CSU}(zt, u)$ , we have  $x\sigma\sigma = x\sigma$  and  $x\sigma\rho = x\theta'$  for  $x = z$  and for all variables  $x$  in  $C$  and  $D$  for some substitution  $\rho$ . Hence,  $\iota$  is the  $\sigma\rho$ -ground instance of  $\iota'$  and therefore liftable.  $\square$

The other SUP inferences might not be liftable, but they are redundant:

**Lemma 28** *Let  $\iota \in GHInf^{gsel}$  be a SUP inference not covered by Lemma 27. Then  $\iota \in GHRed_1^{gsel}(\mathcal{G}(N))$ .*

*Proof* Let  $C\theta = C'\theta \vee s\theta \approx s'\theta$  and  $D\theta = D'\theta \vee t\theta \approx t'\theta$  be the premises of  $\iota$ , where  $s\theta \approx s'\theta$  and  $t\theta \approx t'\theta$  are the literals involved in the inference,  $s\theta \succ s'\theta$ ,  $t\theta \succ t'\theta$ , and  $C'$ ,  $D'$ ,  $s$ ,  $s'$ ,  $t$ ,  $t'$  are the respective subclauses and terms in  $C = \mathcal{G}^{-1}(C\theta)$  and  $D = \mathcal{G}^{-1}(D\theta)$ . Then the inference  $\iota$  looks like this:

$$\frac{D'\theta \vee t\theta \approx t'\theta \quad C'\theta \vee s\theta\langle t\theta \rangle \approx s'\theta}{D'\theta \vee C'\theta \vee s\theta\langle t'\theta \rangle \approx s'\theta} \text{ SUP}$$



To show that  $\iota \in \text{GHRed}_1^{\text{gsel}}(\mathcal{G}(N))$ , it suffices to show that  $\{D \in \mathcal{F}(\mathcal{G}(N)) \mid D \prec \mathcal{F}(C\theta)\} \models \mathcal{F}(\text{concl}(\iota))$ . To this end, let  $\mathcal{J}$  be an interpretation in GF such that  $\mathcal{J} \models \{D \in \mathcal{F}(\mathcal{G}(N)) \mid D \prec \mathcal{F}(C\theta)\}$ . We need to show that  $\mathcal{J} \models \mathcal{F}(\text{concl}(\iota))$ . If  $\mathcal{F}(D'\theta)$  is true in  $\mathcal{J}$ , then obviously  $\mathcal{J} \models \mathcal{F}(\text{concl}(\iota))$ . So we assume that  $\mathcal{F}(D'\theta)$  is false in  $\mathcal{J}$ . Since  $C\theta \succ D\theta$  by the SUP order conditions, it follows that  $\mathcal{J} \models \mathcal{F}(t\theta \approx t'\theta)$ . Therefore, it suffices to show  $\mathcal{J} \models \mathcal{F}(C\theta)$ .

Let  $p$  be the position in  $s\theta$  where the  $\iota$  takes place and  $p'$  be the longest prefix of  $p$  that is a green subterm of  $s$ . Let  $u = s|_{p'}$ . Since Lemma 27 does not apply to  $\iota$ ,  $u$  is not a deep variable; if  $p = p'$ , the variable condition does not hold for  $D$  and  $C$ ; and if  $p \neq p'$ ,  $u$  is a variable. That means either the position  $p$  does not exist in  $s$ , because it is below an unapplied variable that is shallow in  $C$ , or  $s|_p$  is an unapplied variable that is shallow in  $C$  and for which the variable condition does not hold.

CASE 1: The position  $p$  does not exist in  $s$  because it is below a variable  $x$  that is shallow in  $C$ . Then  $t\theta$  is an argument subterm of  $x\theta$  and hence an argument subterm of  $x\theta\bar{w}$  for any arguments  $\bar{w}$ . Let  $v$  be the term that we obtain by replacing  $t\theta$  by  $t'\theta$  in  $x\theta$  at the relevant position. Since  $\mathcal{J} \models \mathcal{F}(t\theta \approx t'\theta)$ , by congruence,  $\mathcal{J} \models \mathcal{F}(x\theta\bar{w} \approx v\bar{w})$  for any arguments  $\bar{w}$ . Hence,  $\mathcal{J} \models \mathcal{F}(C\theta)$  if and only if  $\mathcal{J} \models \mathcal{F}(C\{x \mapsto v\}\theta)$  by congruence. Here, it is crucial that the variable is shallow in  $C$  because congruence does not hold in  $\mathcal{F}$ -encoded terms below  $\lambda$ -binders. By the inference conditions, we have  $t\theta \succ t'\theta$ , which implies  $\mathcal{F}(C\theta) \succ \mathcal{F}(C\{x \mapsto v\}\theta)$  by compatibility with green contexts. Therefore, by the assumption about  $\mathcal{J}$ , we have  $\mathcal{J} \models \mathcal{F}(C\{x \mapsto v\}\theta)$  and hence  $\mathcal{J} \models \mathcal{F}(C\theta)$ .

CASE 2: The term  $s|_p$  is a variable  $x$  that is shallow in  $C$  and for which the variable condition does not hold. From this, we know that  $C\theta \succeq C''\theta$ , where  $C'' = C\{x \mapsto t'\}$ . We cannot have  $C\theta = C''\theta$  because  $x\theta = t\theta \neq t'\theta$  and  $x$  occurs in  $C$ . Hence, we have  $C\theta \succ C''\theta$ . By the definition of  $\mathcal{J}$ ,  $C\theta \succ C''\theta$  implies  $\mathcal{J} \models \mathcal{F}(C''\theta)$ . We will use equalities that are true in  $\mathcal{J}$  to rewrite  $\mathcal{F}(C\theta)$  into  $\mathcal{F}(C''\theta)$ , which implies  $\mathcal{J} \models \mathcal{F}(C\theta)$  by congruence.

By saturation, every ARGCONG inference  $t'$  from  $D$  is in  $\text{HRed}_1^{\text{gsel}}(N)$ —i.e.,  $\mathcal{G}(\text{concl}(t')) \subseteq \mathcal{G}(N) \cup \text{GHRed}_C(\mathcal{G}(N))$ . Hence,  $D'\theta \vee t\theta\bar{u} \approx t'\theta\bar{u}$  is in  $\mathcal{G}(N) \cup \text{GHRed}_C(\mathcal{G}(N))$  for any ground arguments  $\bar{u}$ .

We observe that whenever  $t\theta\bar{u}$  and  $t'\theta\bar{u}$  are smaller than the maximal term of  $C\theta$  for some arguments  $\bar{u}$ , we have

$$\mathcal{J} \models \mathcal{F}(t\theta\bar{u}) \approx \mathcal{F}(t'\theta\bar{u}) \quad (*)$$

To show this, we assume that  $t\theta\bar{u}$  and  $t'\theta\bar{u}$  are smaller than the maximal term of  $C\theta$  and we distinguish two cases: If  $t\theta$  is smaller than the maximal term of  $C\theta$ , all terms in  $D'\theta$  are smaller than the maximal term of  $C\theta$  and hence  $D'\theta \vee t\theta\bar{u} \approx t'\theta\bar{u} \prec C\theta$ . If, on the other hand,  $t\theta$  is equal to the maximal term of  $C\theta$ , then  $t\theta\bar{u}$  and  $t'\theta\bar{u}$  are smaller than  $t\theta$ . Hence  $t\theta\bar{u} \approx t'\theta\bar{u} \prec t\theta \approx t'\theta$  and  $D'\theta \vee t\theta\bar{u} \approx t'\theta\bar{u} \prec D\theta \prec C\theta$ . In both cases, since  $D'\theta$  is false in  $\mathcal{J}$ , by the definition of  $\mathcal{J}$ , we have (\*).

Next, we show the equivalence of  $C\theta$  and  $C''\theta$  via rewriting with equations of the form (\*) where  $t\theta\bar{u}$  and  $t'\theta\bar{u}$  are smaller than the maximal term of  $C\theta$ . Since  $x$  is shallow in  $C$ , every occurrence of  $x$  in  $C$  is not inside a  $\lambda$ -expression and not inside an argument of an applied variable. Therefore, all occurrences of  $x$  in  $C$  are in a green subterm of the form  $x\bar{v}$  for some terms  $\bar{v}$  that do not contain  $x$ . Hence, every occurrence of  $x$  in  $C$  corresponds to a subterm  $\mathcal{F}((x\bar{v})\theta) = \mathcal{F}(t\theta\bar{v}\theta)$  in  $\mathcal{F}(C\theta)$  and to a subterm  $\mathcal{F}((x\bar{v})\{x \mapsto t'\}\theta) = \mathcal{F}(t'\theta\bar{v}\{x \mapsto t'\}\theta) = \mathcal{F}(t'\theta\bar{v}\theta)$  in  $\mathcal{F}(C''\theta)$ . These are the only positions where  $C\theta$  and  $C''\theta$  differ.

To justify the necessary rewrite steps from  $\mathcal{F}(t\theta\bar{v}\theta)$  into  $\mathcal{F}(t'\theta\bar{v}\theta)$  using (\*), we must show that  $\mathcal{F}(t\theta\bar{v}\theta)$  and  $\mathcal{F}(t'\theta\bar{v}\theta)$  are smaller than the maximal term in  $\mathcal{F}(C\theta)$  for the relevant  $\bar{v}$ . If  $\bar{v}$  is the empty tuple, we do not need to show this because  $\mathcal{J} \models \mathcal{F}(t\theta \approx t'\theta)$  follows from

$\mathcal{F}(D\theta)$ 's being true and  $\mathcal{F}(D'\theta)$ 's being false. If  $\bar{v}$  is nonempty, it suffices to show that  $x\bar{v}$  is not a maximal term in  $C$ . Then  $\mathcal{F}(t\theta\bar{v}\theta)$  and  $\mathcal{F}(t'\theta\bar{v}\theta)$ , which correspond to the term  $x\bar{v}$  in  $C$ , cannot be maximal in  $\mathcal{F}(C\theta)$  and  $\mathcal{F}(C''\theta)$  either. Hence they must be smaller than the maximal term in  $\mathcal{F}(C\theta)$  because they are subterms of  $\mathcal{F}(C\theta)$  and  $\mathcal{F}(C''\theta) \prec \mathcal{F}(C\theta)$ , respectively.

To show that  $x\bar{v}$  is not a maximal term in  $C$ , we make a case distinction on whether  $s\theta \approx s'\theta$  is selected in  $C\theta$  or  $s\theta$  is the maximal term in  $C\theta$ . One of these must hold because  $s\theta \approx s'\theta$  is eligible in  $C\theta$ . If it is selected, by the selection restrictions,  $x$  cannot be the head of a maximal term of  $C$ . If  $s\theta$  is the maximal term in  $C\theta$ , we can argue that  $x$  is a green subterm of  $s$  and, since  $x$  is shallow,  $s$  cannot be of the form  $x\bar{v}$  for a nonempty  $\bar{v}$ . This justifies the necessary rewrites between  $\mathcal{F}(C\theta)$  and  $\mathcal{F}(C''\theta)$  and it follows that  $\mathbb{J} \models \mathcal{F}(C\theta)$ .  $\square$

With these properties of our inference systems in place, the saturation framework guarantees static and dynamic refutational completeness of  $HInf_1^{sel}$  w.r.t.  $HRed_1^{sel}$ . However, the framework gives us refutational completeness w.r.t. the Herbrand entailment  $\models_{\mathcal{G}}$ , defined as  $N_1 \models_{\mathcal{G}} N_2$  if  $\mathcal{G}(N_1) \models \mathcal{G}(N_2)$ , whereas our semantics is Tarski entailment  $\models$ —i.e.,  $N_1 \models N_2$  if any model of  $N_1$  is a model of  $N_2$ . To repair this mismatch, we use the following lemma, which can be proved along the lines of Lemma 4.16 of Bentkamp et al. [12].

**Lemma 29** *For  $N \subseteq C_H$ , we have  $N \models_{\mathcal{G}} \perp$  if and only if  $N \models \perp$ .*

**Theorem 30 (Static refutational completeness)** *Let  $N \subseteq C_H$  be a clause set saturated w.r.t.  $HInf_1^{sel}$  and  $HRed_1^{sel}$ . Then  $N \models \perp$  if and only if  $\perp \in N$ .*

*Proof* We apply Theorem 14 of Waldmann et al. We take  $H$  for  $\mathbf{F}$ ,  $\mathcal{G}H$  for  $\mathbf{G}$ , and  $\mathcal{F}(sel)$  for  $\mathcal{Q}$ . It is easy to see that the entailment relation  $\models$  on  $\mathcal{G}H$  is a consequence relation in the sense of the framework. By Lemmas 23 and 24,  $(GHRed_1^{sel}, GHRed_C)$  is a redundancy criterion in the sense of the framework, and  $\mathcal{G}^{gsel}$  are grounding functions in the sense of the framework, for all  $gsel \in \mathcal{F}(sel)$ . The redundancy criterion  $(HRed_1^{sel}, HRed_C)$  matches the intersected lifted redundancy criterion  $Red^{\mathcal{G}, \sqsupset}$  of Waldmann et al. By Theorem 21,  $GHInf_1^{gsel}$  is statically refutationally complete for all  $gsel \in \mathcal{F}(sel)$ . By Lemmas 25, 27, and 28, for every saturated  $N \subseteq C_H$ , there exists a selection function  $gsel \in \mathcal{G}(sel)$  such that all inferences  $\iota \in GHInf_1^{gsel}$  with  $prems(\iota) \in \mathcal{G}(N)$  either are  $\mathcal{G}^{gsel}$ -ground instances of  $HInf_1^{sel}$ -inferences from  $N$  or belong to  $GHRed_1^{gsel}(\mathcal{G}(N))$ .

If  $\sqsupset = \emptyset$ , Theorem 14 of Waldmann et al. implies that if  $N \subseteq C_H$  is a clause set saturated w.r.t.  $HInf_1^{sel}$  and  $HRed_1^{sel}$ , then  $N \models_{\mathcal{G}} \perp$  if and only if  $\perp \in N$ . By Lemma 16 of Waldmann et al., this also holds if  $\sqsupset \neq \emptyset$ . By Lemma 29, this also holds for the Tarski entailment  $\models$ . That is, if  $N \subseteq C_H$  is a clause set saturated w.r.t.  $HInf_1^{sel}$  and  $HRed_1^{sel}$ , then  $N \models \perp$  if and only if  $\perp \in N$ .  $\square$

From static refutational completeness, we can easily derive dynamic refutational completeness. Let  $(N_i)_i$  be a finite or infinite sequence over sets of clauses from  $C_H$ . Such a sequence is called a *derivation* if  $N_i \setminus N_{i+1} \subseteq HRed_C(N_{i+1})$  for all  $i$ . It is called *fair* if all  $HInf$ -inferences from clauses in  $\bigcup_i \bigcap_{j \geq i} N_j$  are contained in  $\bigcup_i HRed_1(N_i)$ .

**Theorem 31 (Dynamic refutational completeness)** *For every fair derivation  $(N_i)_i$  such that  $N_0 \models \perp$ , we have  $\perp \in N_i$  for some  $i$ .*

*Proof* By Theorem 17 of Waldmann et al., this follows from Theorem 30 and Lemma 29.  $\square$

## 5 Extensions

The calculus can be extended to make it more practical. The familiar simplification machinery can be adapted to higher-order terms by considering green contexts instead of arbitrary contexts. Optional inference rules provide lightweight alternatives to the FLUIDSUP rule and the extensionality axiom.

Two of the optional rules below rely on the notion of “orange subterms.” A  $\lambda$ -term  $t$  is an *orange subterm* of a  $\lambda$ -term  $s$  if  $s = t$ ; or if  $s = f(\bar{x})\bar{s}$  and  $t$  is an orange subterm of  $s_i$  for some  $i$ ; or if  $s = x\bar{s}$  and  $t$  is an orange subterm of  $s_i$  for some  $i$ ; or if  $s = (\lambda x. u)$  and  $t$  is an orange subterm of  $u$ . In the term  $f(g a)(y b)(\lambda x. h c(g x))$ , the orange subterms are all the green subterms— $a, g a, y b, \lambda x. h c(g x)$  and the whole term—and in addition  $b, c, x, g x$ , and  $h c(g x)$ . Following the convention introduced in Sect. 2, this notion is lifted to  $\beta\eta$ -equivalence classes via representatives in  $\eta$ -short  $\beta$ -normal form. We write  $t = s \langle\langle \bar{x}_n. u \rangle\rangle$  to indicate that  $u$  is an orange subterm of  $t$ , where  $\bar{x}_n$  are the variables bound in the *orange context* around  $u$ , from outermost to innermost. If  $n = 0$ , we simply write  $t = s \langle\langle u \rangle\rangle$ .

Once a term  $s \langle\langle \bar{x}_n. u \rangle\rangle$  has been introduced, we write  $s \langle\langle \bar{x}_n. u' \rangle\rangle_\eta$  to denote the same context with a different subterm  $u'$  at that position. The  $\eta$  subscript is a reminder that  $u'$  is not necessarily an orange subterm of  $s \langle\langle \bar{x}_n. u' \rangle\rangle_\eta$  due to potential applications of  $\eta$ -reduction. For example, if  $s \langle\langle x. g x x \rangle\rangle = h a(\lambda x. g x x)$ , then  $s \langle\langle x. f x \rangle\rangle_\eta = h a(\lambda x. f x) = h a f$ .

Demodulation, which destructively rewrites using an equality  $t \approx t'$ , is available at green positions. A variant rewrites inside  $\lambda$ -expressions:

$$\frac{t \approx t' \quad C \langle\langle s \langle\langle \bar{x}. t\sigma \rangle\rangle \rangle}{t \approx t' \quad C \langle\langle s \langle\langle \bar{x}. t'\sigma \rangle\rangle_\eta \rangle \quad s \langle\langle \bar{x}. t\sigma \rangle\rangle \approx s \langle\langle \bar{x}. t'\sigma \rangle\rangle_\eta} \lambda\text{DEMOMODEXT}$$

where  $s \langle\langle \bar{x}. t\sigma \rangle\rangle \downarrow_{\beta\eta}$  is a  $\lambda$ -expression or an applied variable. The term  $t\sigma$  may refer to the bound variables  $\bar{x}$ . There are two side conditions:  $s \langle\langle \bar{x}. t\sigma \rangle\rangle \succ s \langle\langle \bar{x}. t'\sigma \rangle\rangle_\eta$  and the second premise is larger ( $\succ$ ) than the third conclusion. This ensures that this premise is redundant w.r.t. these conclusions and may be removed. The double bar indicates that the conclusions collectively make the premises redundant and can replace them.

The third conclusion, which is entailed by  $t \approx t'$  and (EXT), could be safely omitted if the corresponding (EXT) instance, with  $y := (\lambda x. s \langle\langle \bar{x}. t\sigma \rangle\rangle)$  and  $z := (\lambda x. s \langle\langle \bar{x}. t'\sigma \rangle\rangle_\eta)$ , is smaller than the second premise. But in general, it would appear that the third conclusion is necessary and that the variant of  $\lambda\text{DEMOMODEXT}$  that omits it—let us call it  $\lambda\text{DEMOMOD}$ —does not preserve refutational completeness.

An instance of  $\lambda\text{DEMOMODEXT}$ , where  $g z$  is rewritten to  $f z z$  under a  $\lambda$ -binder, follows:

$$\frac{g x \approx f x x \quad k(\lambda z. h(g z)) \approx c}{g x \approx f x x \quad k(\lambda z. h(f z z)) \approx c \quad (\lambda z. h(g z)) \approx (\lambda z. h(f z z))} \lambda\text{DEMOMODEXT}$$

**Lemma 32**  $\lambda\text{DEMOMODEXT}$  is sound and preserves refutational completeness of the calculus.

*Proof* Soundness of the first conclusion is obvious. Soundness of the second and third conclusion follows from congruence and extensionality using the premises. Preservation of completeness is justified by redundancy. Specifically, we justify the deletion of the second premise by showing that it is redundant w.r.t. the conclusions. By definition, it is redundant if for every ground instance  $C \langle\langle s \langle\langle \bar{x}. t\sigma \rangle\rangle \rangle \theta \in \mathcal{G}(C \langle\langle s \langle\langle \bar{x}. t\sigma \rangle\rangle \rangle)$ , its encoding  $\mathcal{F}(C \langle\langle s \langle\langle \bar{x}. t\sigma \rangle\rangle \rangle \theta)$  is entailed by  $\mathcal{F}(\mathcal{G}(N))$ , where  $N$  are the conclusions of  $\lambda\text{DEMOMODEXT}$ .

The first conclusion cannot help us prove redundancy because  $s \ll \bar{x}. t\sigma \gg \theta \downarrow_{\beta\eta}$  might be a  $\lambda$ -expression and then  $\mathcal{F}(s \ll \bar{x}. t\sigma \gg \theta)$  is a symbol that is unrelated to  $\mathcal{F}(t\sigma\theta)$ . Instead, we use the  $\theta$ -instances of the last two conclusions. By Lemma 9,  $\mathcal{F}(C \langle s \ll \bar{x}. t\sigma \gg \eta \rangle \theta)$  has  $\mathcal{F}(s \ll \bar{x}. t\sigma \gg \eta \theta)$  as a subterm. If this subterm is replaced by  $\mathcal{F}(s \ll \bar{x}. t\sigma \gg \theta)$ , we obtain  $\mathcal{F}(C \langle s \ll \bar{x}. t\sigma \gg \theta \rangle)$ . Hence, the  $\mathcal{F}$ -encodings of the  $\theta$ -instances of the last two conclusions entail the  $\mathcal{F}$ -encoding of the  $\theta$ -instance of the second premise by congruence. Due to the side condition that the second premise is larger than the second and third conclusion, by stability under grounding substitutions, the  $\theta$ -instances of the last two conclusions must be smaller than the  $\theta$ -instance of the second premise. Thus, the second premise is redundant.  $\square$

The next simplification rule can be used to prune arguments of applied variables if the arguments can be expressed as functions of the remaining arguments. For example, the clause  $C[y \text{ a b } (f \text{ b a}), y \text{ b d } (f \text{ d b})]$ , in which  $y$  occurs twice, can be simplified to  $C[y' \text{ a b }, y' \text{ b d}]$ . Here, for each occurrence of  $y$ , the third argument can be computed by applying  $f$  to the second and first arguments. The rule can also be used to remove the repeated arguments in  $y \text{ b b } \not\approx y \text{ a a}$ , the static argument  $a$  in  $y \text{ a c } \not\approx y \text{ a b}$ , and all four arguments in  $y \text{ a b } \not\approx z \text{ b d}$ . It is stated as

$$\frac{C}{C\sigma} \text{ PRUNEARG}$$

where  $\sigma = \{y \mapsto \lambda \bar{x}_j. y' \bar{x}_{j-1} \}$ ,  $y'$  is a fresh variable,  $C \sqsupset C\sigma$ , the minimum number  $k$  of arguments passed to any occurrence of  $y$  in the clause  $C$  is at least  $j$ , and there exists a term  $t$  containing no variables bound in the clause such that  $s_j = t \bar{s}_{j-1} s_{j+1} \dots s_k$  for all terms of the form  $y \bar{s}_k$  occurring in the clause.

Clauses with a static argument correspond to the case  $t := (\lambda \bar{x}_{j-1} x_{j+1} \dots x_k. u)$ , where  $u$  is the static argument (containing no variables bound in  $t$ ) and  $j$  is its index in  $y$ 's argument list. The repeated argument case corresponds to  $t := (\lambda \bar{x}_{j-1} x_{j+1} \dots x_k. x_i)$ , where  $i$  is the index of the repeated argument's mate.

**Lemma 33** *PRUNEARG is sound and preserves refutational completeness of the calculus.*

*Proof* The rule is sound because it simply applies a substitution to  $C$ . It preserves completeness because the premise  $C$  is redundant w.r.t. the conclusion  $C\sigma$ . This is because the sets of ground instances of  $C$  and  $C\sigma$  are the same and  $C \sqsupset C\sigma$ . Clearly  $C\sigma$  is an instance of  $C$ . We will show the inverse: that  $C$  is an instance of  $C\sigma$ . Let  $\rho = \{y' \mapsto \lambda \bar{x}_{j-1} x_{j+1} \dots x_k. y \bar{x}_{j-1} (t \bar{x}_{j-1} x_{j+1} \dots x_k) x_{j+1} \dots x_k\}$ . We show  $C\sigma\rho = C$ . Consider an occurrence of  $y$  in  $C$ . By the side conditions, it will have the form  $y \bar{s}_k \bar{u}$ , where  $s_j = t \bar{s}_{j-1} s_{j+1} \dots s_k$ . Hence,  $(y \bar{s}_k)\sigma\rho = (y' \bar{s}_{j-1} s_{j+1} \dots s_k)\rho = y \bar{s}_{j-1} (t \bar{s}_{j-1} s_{j+1} \dots s_k) s_{j+1} \dots s_k = y \bar{s}_k$ . Thus,  $C\sigma\rho = C$ .  $\square$

We designed an algorithm that efficiently computes the subterm  $u$  of the term  $t = (\lambda x_1 \dots x_{j-1} x_{j+1} \dots x_k. u)$  occurring in the side conditions of PRUNEARG. The algorithm is incomplete, but our tests suggest that it discovers most cases of prunable arguments that occur in practice. The algorithm works by maintaining a mapping of pairs  $(y, i)$  of functional variables  $y$  and indices  $i$  of their arguments to a set of candidate terms for  $u$ . For an occurrence  $y \bar{s}_n$  of  $y$  and for an argument  $s_j$ , the algorithm approximates this set by computing all possible ways in which subterms of  $s_j$  that are equal to any other  $s_i$  can be replaced with the variable  $x_i$  corresponding to the  $i$ th argument of  $y$ . The candidate sets for all occurrences of  $y$  are then intersected. An arbitrary element of the final intersection is returned as the term  $u$ .

For example, suppose that  $y \text{ a } (f \text{ a}) \text{ b}$  and  $y \text{ z } (f \text{ z}) \text{ b}$  are the only occurrences of  $y$  in the clause  $C$ . The initial mapping is  $\{1 \mapsto \mathcal{T}_H, 2 \mapsto \mathcal{T}_H, 3 \mapsto \mathcal{T}_H\}$ . After computing the ways in

which each argument can be expressed using the remaining ones for the first occurrence and intersecting the sets, we get  $\{1 \mapsto \{a\}, 2 \mapsto \{f a, f x_1\}, 3 \mapsto \{b\}\}$ , where  $x_1$  represents  $y$ 's first argument. Finally, after computing the corresponding sets for the second occurrence of  $y$  and intersecting them with the previous candidate sets, we get  $\{1 \mapsto \emptyset, 2 \mapsto \{f x_1\}, 3 \mapsto \{b\}\}$ . The final mapping shows that we can remove the second argument, since it can be expressed as a function of the first argument:  $t = (\lambda x_1 x_3. f x_1 x_3)$ . We can also remove the third argument, since its value is fixed:  $t = (\lambda x_1 x_3. b)$ . An example where our procedure fails is the pair of occurrences  $y(\lambda x. a)(f a) c$  and  $y(\lambda x. b)(f b) d$ . PRUNEARG can be used to eliminate the second argument by taking  $t := (\lambda x_1 x_3. f(x_1 x_3))$ , but our algorithm will not detect this.

Following the literature [36, 61], we provide a rule for negative extensionality:

$$\frac{C' \vee s \not\approx s'}{C' \vee s(\text{sk}(\bar{a})\bar{y}) \not\approx s'(\text{sk}(\bar{a})\bar{y})} \text{NEGEXT}$$

where  $\text{sk}$  is a fresh Skolem symbol,  $\bar{a}$  and  $\bar{y}$  are the type and term variables occurring free in the the literal  $s \not\approx s'$ , and  $s \not\approx s'$  is eligible in the premise. Negative extensionality can also be applied as a simplification rule to all literals in the initial problem. The rule uses Skolem terms  $\text{sk} \bar{y}$  rather than  $\text{diff } s s'$  because they tend to be more compact.

**Lemma 34 (NEGEXT's satisfiability preservation)** *Let  $N \subseteq C_H$  and let  $E$  be the conclusion of a NEGEXT inference from  $N$ . If  $N \cup \{(\text{EXT})\}$  is satisfiable, then  $N \cup \{(\text{EXT}), E\}$  is satisfiable.*

*Proof* Let  $\mathcal{J}$  be a model of  $N \cup \{(\text{EXT})\}$ . We need to construct a model of  $N \cup \{(\text{EXT}), E\}$ . Since (EXT) holds in  $\mathcal{J}$ , so does its instance  $s(\text{diff } s s') \not\approx s'(\text{diff } s s') \vee s \approx s'$ , where  $\tau \rightarrow \nu$  is the type of  $s$  and  $s'$ . We extend the model  $\mathcal{J}$  to a model  $\mathcal{J}'$ , interpreting  $\text{sk}$  such that  $\mathcal{J}' \models \text{sk}(\bar{a})\bar{y} \approx \text{diff } s s'$  for each  $i$ . The Skolem symbol  $\text{sk}$  takes the free type and term variables of  $s \not\approx s'$  as arguments, which include all the free variables of  $\text{diff } s s'$ , allowing us to extend  $\mathcal{J}$  in this way.

By assumption, the premise  $C' \vee s \not\approx s'$  is true in  $\mathcal{J}$  and hence in  $\mathcal{J}'$ . Since the above instance of (EXT) holds in  $\mathcal{J}$ , it also holds in  $\mathcal{J}'$ . Hence, the conclusion  $C' \vee s(\text{sk}(\bar{a}_m)\bar{y}_n) \not\approx s'(\text{sk}(\bar{a}_m)\bar{y}_n)$  also holds, which can be seen by resolving the premise against the (EXT) instance and unfolding the defining equation of  $\text{sk}$ .  $\square$

One reason why the extensionality axiom is so prolific is that both sides of its maximal literal,  $y(\text{diff } yz) \not\approx z(\text{diff } yz)$ , are fluid. As a pragmatic alternative to the axiom, we introduce the “abstracting” rules ABSSUP, ABSEQRES, and ABSEQFACT with the same premises as the core SUP, EQRES, and EQFACT, respectively. We call these rules collectively ABS. Each new rule shares all the side conditions of the corresponding core rule except that of the form  $\sigma \in \text{CSU}(s, t)$ . Instead, it lets  $\sigma$  be the most general unifier of  $s$  and  $t$ 's types and adds this condition: Let  $\nu \langle s_1, \dots, s_n \rangle = s\sigma$  and  $\nu \langle t_1, \dots, t_n \rangle = t\sigma$ , where  $\nu \langle \rangle$  is the largest common green context of  $s\sigma$  and  $t\sigma$ . If any  $s_i$  is of functional type and the core rule has conclusion  $E\sigma$ , the new rule has conclusion  $E\sigma \vee s_1 \not\approx t_1 \vee \dots \vee s_n \not\approx t_n$ . The NEGEXT rule can then be applied to those literals  $s_i \not\approx t_i$  whose sides have functional type. Essentially the same idea was proposed by Bhayat and Reger as *unification with abstraction* in the context of combinatory superposition [20, Sect. 3.1]. The approach regrettably does not fully eliminate the need for axiom (EXT), as Visa Nummelin demonstrated via the following example.

**Example 35** Consider the unsatisfiable clause set consisting of  $h x \approx f x$ ,  $kh \approx k g$ , and  $kg \not\approx kf$ , where  $k$  takes at most one argument and  $h \succ g \succ f$ . The only nonredundant ABS inference applicable is ABSEQRES on the third clause, resulting in  $g \not\approx f$ . Applying EXTNEG further produces  $g \text{sk} \not\approx f \text{sk}$ . The set consisting of all five clauses is saturated.

A different approach is to instantiate the extensionality axiom with arbitrary terms  $s, s'$  of the same functional type, which presumably appear as green subterms in the current clause set:

$$\frac{}{s(\text{diff } s s') \not\approx s'(\text{diff } s s') \vee s \approx s'} \text{EXTINST}$$

Intuitively, if we think in terms of eligibility, EXTINST demands  $s(\text{diff } s s') \approx s'(\text{diff } s s')$  to be proved before  $s \approx s'$  can be used. This can be advantageous because simplifying inferences (based on matching) will often be able to rewrite the applied terms  $s(\text{diff } s s')$  and  $s'(\text{diff } s s')$ . In contrast, ABS assume  $s \approx s'$  and delay the proof obligation that  $s(\text{diff } s s') \approx s'(\text{diff } s s')$ . This can create many long clauses, which will be subject to expensive generating inferences (based on full unification).

Superposition can be generalized to orange subterms as follows:

$$\frac{D' \vee t \approx t' \quad C' \vee s \ll \bar{x}. u \gg \approx s'}{(D' \vee C' \vee s \ll \bar{x}. t' \gg)_\eta \approx s'} \lambda\text{SUP}$$

SUP's side conditions apply. We also require that  $\bar{x}$  has length  $n > 0$ ,  $\bar{x}\sigma = \bar{x}$ , and the variables  $\bar{x}$  do not occur in  $y\sigma$  for all variables  $y$  in  $u$ . Moreover, let  $P_y = \{y\}$  for all type and term variables  $y \notin \bar{x}$ . For each  $i$ , let  $P_{x_i}$  be recursively defined as the union of all  $P_y$  such that  $y$  occurs free in the  $\lambda$ -expression that binds  $x_i$  in  $s \ll \bar{x}. u \gg \sigma$  or that occurs free in the corresponding subterm of  $s \ll \bar{x}. t' \gg_\eta \sigma$ . The substitution  $\rho$  is defined as  $\{x_i \mapsto \text{sk}_i \langle \bar{\alpha}_i \rangle \bar{y}_i\}$  for each  $i$ , where  $\bar{y}_i$  are the term variables in  $P_{x_i}$  and  $\bar{\alpha}_i$  are the type variables in  $P_{x_i}$  and the type variables occurring in the type of the  $\lambda$ -expression binding  $x_i$ . This substitution introduces Skolem terms to represent bound variables that would otherwise escape their binders. The rule can be justified in terms of paramodulation and extensionality, with the Skolem terms standing for diff terms. An instance of the rule follows:

$$\frac{n \approx \text{zero} \vee \text{div } n \approx \text{one} \quad \text{prod } K (\lambda k. \text{div} (\text{succ } k) (\text{succ } k)) \not\approx \text{one}}{\text{succ } \text{sk} \approx \text{zero} \vee \text{prod } K (\lambda k. \text{one}) \not\approx \text{one}} \lambda\text{SUP}$$

(Intuitively, the term  $\text{prod } K (\lambda k. u)$  is intended to denote the product  $\prod_{k \in K} u$ , where  $k$  ranges over a finite set  $K$  of natural numbers.)

**Lemma 36 ( $\lambda\text{SUP}$ 's satisfiability preservation)** *Let  $N \subseteq C_H$  and let  $E$  be the conclusion of a  $\lambda\text{SUP}$  inference from  $N$ . If  $N \cup \{(\text{EXT})\}$  is satisfiable, then  $N \cup \{(\text{EXT}), E\}$  is satisfiable.*

*Proof* Let  $\mathcal{J}$  be a model of  $N \cup \{(\text{EXT})\}$ . We need to construct a model of  $N \cup \{(\text{EXT}), E\}$ . For each  $i$ , let  $v_i$  be the  $\lambda$ -expression binding  $x_i$  in the term  $s \ll \bar{x}. u \gg \sigma$  in the rule. Let  $v'_i$  be the variant of  $v_i$  in which the relevant occurrence of  $u\sigma$  is replaced by  $t'\sigma$ . We define a substitution  $\pi$  recursively by  $x_i\pi = \text{diff} (v_i\pi) (v'_i\pi)$  for all  $i$ . This definition is well founded because the variables  $x_j$  with  $j \geq i$  do not occur freely in  $v_i$  and  $v'_i$ . We extend the model  $\mathcal{J}$  to a model  $\mathcal{J}'$ , interpreting  $\text{sk}_i$  such that  $\mathcal{J}' \models \text{sk}_i \langle \bar{\alpha}_i \rangle \bar{y}_i \approx \text{diff} (v_i\pi) (v'_i\pi)$  for each  $i$ . Since the free type and term variables of any  $x_i\pi$  are necessarily contained in  $P_{x_i}$ , the arguments of  $\text{sk}_i$  include the free variables of  $\text{diff} (v_i\pi) (v'_i\pi)$ , allowing us to extend  $\mathcal{J}$  in this way.

By assumption, the premises of the  $\lambda\text{SUP}$  inference are true in  $\mathcal{J}$  and hence in  $\mathcal{J}'$ . We need to show that the conclusion  $(D' \vee C' \vee s \ll \bar{x}. t' \gg)_\eta \approx s'$  is also true in  $\mathcal{J}'$ . Let  $\xi$  be a valuation. If  $\mathcal{J}', \xi \models (D' \vee C')\sigma\rho$ , we are done. So we assume that  $D'\sigma\rho$  and  $C'\sigma\rho$  are false in  $\mathcal{J}'$  under  $\xi$ . In the following, we omit ' $\mathcal{J}', \xi \models$ ', but all equations ( $\approx$ ) are meant to be true in  $\mathcal{J}'$  under  $\xi$ . Assuming  $D'\sigma\rho$  and  $C'\sigma\rho$  are false, we will show inductively that  $v_i\pi \approx v'_i\pi$  for all  $i = k, \dots, 1$ . By this assumption, the premises imply that  $t\sigma\rho \approx t'\sigma\rho$  and

$s\langle\langle\bar{x}.u\rangle\rangle\sigma\rho \approx s'\sigma\rho$ . Due to the way we constructed  $\mathcal{J}'$ , we have  $w\pi \approx w\rho$  for any term  $w$ . Hence, we have  $t\sigma\pi \approx t'\sigma\pi$ . The terms  $v_k\pi(\text{diff}(v_k\pi)(v'_k\pi))$  and  $v'_k\pi(\text{diff}(v_k\pi)(v'_k\pi))$  are the respective result of applying  $\pi$  to the body of the  $\lambda$ -expressions  $v_k$  and  $v'_k$ . Therefore, by congruence,  $t\sigma\pi \approx t'\sigma\pi$  and  $t\sigma = u\sigma$  imply that  $v_k\pi(\text{diff}(v_k\pi)(v'_k\pi)) \approx v'_k\pi(\text{diff}(v_k\pi)(v'_k\pi))$ . The extensionality axiom then implies  $v_k\pi \approx v'_k\pi$ .

It follows directly from the definition of  $\pi$  that for all  $i$ ,  $v_i\pi(\text{diff}(v_i\pi)(v'_i\pi)) = s_i\langle\langle v_{i+1}\pi\rangle\rangle$  and  $v'_i\pi(\text{diff}(v_i\pi)(v'_i\pi)) = s_i\langle\langle v'_{i+1}\pi\rangle\rangle$  for some context  $s_i\langle\langle \rangle\rangle$ . The subterms  $v_{i+1}\pi$  of  $s_i\langle\langle v_{i+1}\pi\rangle\rangle$  and  $v'_{i+1}\pi$  of  $s_i\langle\langle v'_{i+1}\pi\rangle\rangle$  may be below applied variables but not below  $\lambda$ s. Since substitutions avoid capture, in  $v_i$  and  $v'_i$ ,  $\pi$  only substitutes  $x_j$  with  $j < i$ , but in  $v_{i+1}$  and  $v'_{i+1}$ , it substitutes all  $x_j$  with  $j \leq i$ . By an induction using these equations, congruence, and the extensionality axiom, we can derive from  $v_k\pi \approx v'_k\pi$  that  $v_1\pi \approx v'_1\pi$ . Since  $\mathcal{J}' \models w\pi \approx w\rho$  for any term  $w$ , we have  $v_1\rho \approx v'_1\rho$ . By congruence, it follows that  $s\langle\langle\bar{x}.u\rangle\rangle\sigma\rho \approx s\langle\langle\bar{x}.t'\rangle\rangle_\eta\sigma\rho$ . With  $s\langle\langle\bar{x}.u\rangle\rangle\sigma\rho \approx s'\sigma\rho$ , it follows that  $(s\langle\langle\bar{x}.t'\rangle\rangle_\eta \approx s')\sigma\rho$ . Hence, the conclusion of the  $\lambda$ SUP inference is true in  $\mathcal{J}'$ .  $\square$

The next rule, *duplicating flex subterm superposition*, is a lightweight alternative to FLUIDSUP:

$$\frac{D' \vee t \approx t' \quad C' \vee s\langle y \bar{u}_n \rangle \approx s'}{(D' \vee C' \vee s\langle z \bar{u}_n t' \rangle \approx s')\rho\sigma} \text{DUPSUP}$$

where  $n > 0$ ,  $\rho = \{y \mapsto \lambda \bar{x}_n. z \bar{x}_n (w \bar{x}_n)\}$ , and  $\sigma \in \text{CSU}(t, w(\bar{u}_n\rho))$  for fresh variables  $w, z$ . The order and eligibility restrictions are as for SUP. The rule can be understood as the composition of an inference that applies the substitution  $\rho$  and of a paramodulation inference into the subterm  $w(\bar{u}_n\rho)$  of  $s\langle z(\bar{u}_n\rho)(w(\bar{u}_n\rho)) \rangle$ . DUPSUP is general enough to replace FLUIDSUP in Examples 2 and 3 but not in Example 4. On the other hand, FLUIDSUP's unification problem is usually a flex–flex pair, whereas DUPSUP yields a less explosive flex–rigid pair unless  $t$  is an applied variable.

The last rule, *flex subterm superposition*, is an even more lightweight alternative to FLUIDSUP:

$$\frac{D' \vee t \approx t' \quad C' \vee s\langle y \bar{u}_n \rangle \approx s'}{(D' \vee C' \vee s\langle t' \rangle \approx s')\sigma} \text{FLEXSUP}$$

where  $n > 0$  and  $\sigma \in \text{CSU}(t, y\bar{u}_n)$ . The order and eligibility restrictions are as for SUP.

## 6 Implementation

Zipperposition [29,30] is an open source superposition prover written in OCaml.<sup>1</sup> Originally designed for polymorphic first-order logic (TF1 [22]), it was later extended with an incomplete higher-order mode based on pattern unification [53]. Bentkamp et al. [13] extended it further with a complete  $\lambda$ -free clausal higher-order mode. We have now implemented a clausal higher-order mode based on our calculus. We use the order  $\succ_\lambda$  (Sect. 3.5) derived from a  $\lambda$ -free Knuth–Bendix order [9] with the precedence  $\text{lam} \succ \dots \succ \text{db}_k \succ \dots \succ \text{db}_1 \succ \text{db}_0$ . We currently use the corresponding nonstrict order  $\succeq_\lambda$  as  $\succsim$ .

Except for FLUIDSUP, the core calculus rules already existed in Zipperposition in a similar form. To improve efficiency, we added appropriate indexes for all new inference rules that have two premises. Among the optional rules, we implemented  $\lambda$ DEMOM, PRUNE-ARG, NEGEXT, ABS, EXTINST,  $\lambda$ SUP, DUPSUP, and FLEXSUP. For  $\lambda$ DEMOM and  $\lambda$ SUP,

<sup>1</sup> <https://github.com/sneeuwballen/zipperposition>

demodulation, subsumption, and other standard simplification rules (as implemented in E [58]), we use pattern unification. For generating inference rules that require enumerations of complete sets of unifiers, we use the complete procedure of Vukmirović et al. [65].

Zipperposition implements a DISCOUNT-style given clause procedure [5]. The proof state is represented by a set  $A$  of active clauses and a set  $P$  of passive clauses. To interleave nonterminating unification with other computation, we added a set  $T$  containing possibly infinite sequences of scheduled inferences. These sequences are stored as finite instructions of how to compute the inferences. Initially, all clauses are in the passive set. At each iteration of the main loop, the prover heuristically selects a *given clause*  $C$  from  $P$ . If  $P$  is empty, sequences from  $T$  are evaluated to generate more clauses into  $P$ . Once it is selected,  $C$  is simplified using  $A$ . Clauses in  $A$  are simplified w.r.t.  $C$  and moved to  $P$  if simplified. Then  $C$  is added to  $A$  and all nonredundant inferences between  $C$  and  $A$  are scheduled into  $T$ . This maintains the invariant that all nonredundant inferences between clauses in  $A$  have been scheduled or performed. Then some of the scheduled inferences in  $T$  are performed and the conclusions are put into  $P$ . Finally the loop restarts.

We can view the above loop as an instance of the abstract Zipperposition loop prover ZL of Waldmann et al. [67, Example 34]. Their Theorem 32 allows us to obtain dynamic completeness for this prover architecture from our static completeness result (Theorem 30). This requires that the sequences in  $T$  are visited fairly, that clauses in  $P$  are chosen fairly, and that simplification terminates, all of which are guaranteed by our implementation.

The unification procedure we use returns a sequence of either singleton sets containing the unifier or an empty set signaling that a unifier is still not found. Empty sets are returned to give back control to the caller of unification procedure and avoid getting stuck on nonterminating problems. These sequences of unifier subsingletons are converted into sequences containing subsingletons of clauses representing inference conclusions.

## 7 Evaluation

We evaluated our prototype implementation of the calculus in Zipperposition with other higher-order provers and with Zipperposition’s modes for less expressive logics. All of the experiments were performed on StarExec nodes equipped with Intel Xeon E5-2609 0 CPUs clocked at 2.40 GHz. Following CASC-27,<sup>2</sup> we use 180 s as the CPU time limit.

We used both standard TPTP benchmarks [62] and Sledgehammer-generated benchmarks [52]. From the TPTP, version 7.2.0, we randomly selected 1000 first-order (FO) problems in CNF, FOF, or TFF syntax without arithmetic and all 499 monomorphic higher-order theorems in TH0 syntax without interpreted Booleans and arithmetic. We partitioned the TH0 problems into those containing no  $\lambda$ -expressions (TH0 $\lambda$ f, 452 problems) and those containing  $\lambda$ -expressions (TH0 $\lambda$ , 47 problems). The Sledgehammer benchmarks, corresponding to Isabelle’s Judgment Day suite [25], were regenerated to target clausal higher-order logic. They comprise 2506 problems, divided in two groups: SH- $\lambda$  preserves  $\lambda$ -expressions, whereas SH-ll encodes them as  $\lambda$ -lifted supercombinators [52] to make the problems accessible to  $\lambda$ -free clausal higher-order provers. Each group of problems is generated from 256 Isabelle facts (definitions and lemmas). Our results are publicly available.<sup>3</sup>

**Evaluation of Extensions** To assess the usefulness of the extensions described in Sect. 5, we fixed a *base* configuration of Zipperposition parameters. For each extension, we then

<sup>2</sup> <http://tptp.cs.miami.edu/CASC/>

<sup>3</sup> [http://matryoshka.gforge.inria.fr/pubs/lamsup\\_article\\_results.zip](http://matryoshka.gforge.inria.fr/pubs/lamsup_article_results.zip)



	-NE,-PA	-NE	-PA	Base
TH0	446 (0)	446 (0)	447 (0)	447 (0)
SH- $\lambda$	431 (0)	433 (0)	433 (0)	436 (1)

**Fig. 1** Number of problems proved without rules included in the base configuration

	Base	+ $\lambda$ D	+ $\lambda$ S0	+ $\lambda$ S1	+ $\lambda$ S2	+ $\lambda$ S4	+ $\lambda$ S8	+ $\lambda$ S1024
TH0	447 (0)	448 (0)	449 (0)	449 (0)	449 (0)	449 (0)	449 (0)	449 (0)
SH- $\lambda$	436 (1)	435 (4)	430 (1)	429 (0)	429 (0)	429 (0)	429 (0)	429 (0)

**Fig. 2** Number of problems proved using rules that perform rewriting under  $\lambda$ -binders

	Base	+ABS	+EXTINST	+(EXT)
TH0	447 (0)	450 (1)	450 (1)	376 (0)
SH- $\lambda$	436 (11)	430 (11)	402 (1)	364 (2)

**Fig. 3** Number of problems proved using rules that perform extensionality reasoning

	-FLEXSUP	Base	-FLEXSUP,+DUPSUP	-FLEXSUP,+FLUIDSUP
TH0	446 (0)	447 (0)	448 (1)	447 (0)
SH- $\lambda$	469 (10)	436 (4)	451 (3)	461 (7)

**Fig. 4** Number of problems proved with rules that perform superposition into fluid terms

changed the corresponding parameters and observed the effect on the success rate. The base configuration uses the complete variant of the unification procedure of Vukmirović et al. [65]. It also includes the optional rules NEGEXT and PRUNEARG, substitutes FLEXSUP for the highly explosive FLUIDSUP, and excludes axiom (EXT). The base configuration is not refutationally complete.

The rules NEGEXT (NE) and PRUNEARG (PA) were added to the base configuration because our informal experiments showed that they usually help. Figure 1 confirms this, although the effect is small. In the figures,  $+R$  denotes the inclusion of a rule  $R$  not present in the base, and  $-R$  denotes the exclusion of a rule  $R$  present in the base. Numbers given in parentheses denote the number of problems that are solved only by given configuration and no other configuration in the same figure.

The rules  $\lambda$ DEMODO ( $\lambda$ D) and  $\lambda$ SUP extend the calculus to perform some rewriting under  $\lambda$ -binders. While experimenting with the calculus, we noticed that, in some configurations,  $\lambda$ SUP performs better when the number of fresh Skolem symbols it introduces overall is bounded by some parameter  $n$ . As Fig. 2 shows, inclusion of these rules has different effect on the two benchmark sets. Different choices of  $n$  for  $\lambda$ SUP (denoted by  $\lambda$ S $n$ ) do not seem to influence the success rate much.

The evaluation of the ABS and EXTINST rules and axiom (EXT), presented in Fig. 3, confirms our intuition that including the extensionality axiom is severely detrimental to performance. The raw data reveal that there are no cases in which the axiom helped solve a problem that could not be solved by using one of the two rules instead, even though the  $+(EXT)$  configuration solved 2 unique problems on SH- $\lambda$  benchmarks.

The FLEXSUP rule included in the base configuration did not perform as well as we expected. Even the FLUIDSUP and DUPSUP rules outperformed FLEXSUP, as shown in Fig. 4. This effect is especially visible on SH- $\lambda$  benchmarks. On TPTP, the differences are negligible.

**Main Evaluation** We selected all contenders in the THF division of CASC-27 as representatives of the state of the art: CVC4 1.8 prerelease [8], Leo-III 1.4 [61], Satallax 3.4 [26], and Vampire 4.4 [19]. We also included Ehoh [66], the  $\lambda$ -free clausal higher-order mode of E 2.4. Leo-III and Satallax are cooperative higher-order provers that can be set up to regularly invoke first-order provers as terminal proof procedures. To assess the performance of their core calculi, we evaluated them with first-order backends disabled. We denote these “uncooperative” configurations by Leo-III-uncoop and Satallax-uncoop respectively, as opposed to the standard versions Leo-III-coop and Satallax-coop.

To evaluate the overhead our calculus incurs on first-order or  $\lambda$ -free higher-order problems, we ran Zipperposition in first-order (FOZip) and  $\lambda$ -free ( $\lambda$ freeZip) modes, as well as in a mode that performs the applicative encoding before using first-order Zipperposition (@+FOZip). We evaluated the implementation of our calculus in Zipperposition ( $\lambda$ Zip) in three configurations: base, pragmatic, and full. Pragmatic builds on base by disabling FLEXSUP and replacing complete unification with the pragmatic variant procedure  $pv_{1121}^2$  of Vukmirović et al. Full is a refutationally complete extension of base that substitutes FLUIDSUP for FLEXSUP and includes axiom (EXT). Finally, we evaluated Zipperposition in a portfolio mode that runs the prover in various configurations (Zip-uncoop). We also evaluated a cooperative version of the portfolio that uses Ehoh as backend on higher-order problems (Zip-coop). On first-order problems, we ran Ehoh, Vampire, and Zip-uncoop using the provers’ respective first-order modes.

A summary of these experiments is presented in Fig. 5. In the pragmatic configuration, our calculus outperformed  $\lambda$ freeZip on TH0 $\lambda$ f problems and incurred less than 1% overhead compared with FOZip, but fell behind  $\lambda$ freeZip on SH-II problems. The full configuration suffers greatly from the explosive extensionality axiom and FLUIDSUP rule.

Both base and pragmatic configurations outperformed Leo-III-uncoop, which runs a fixed configuration, by a substantial margin. Similarly, Zip-uncoop outperformed Satallax-uncoop, which runs a portfolio of configurations. Our most competitive configuration, Zip-coop, emerges as the winner on both problem sets containing  $\lambda$ -expressions.

	FO	TH0 $\lambda$ f	TH0 $\lambda$	SH-II	SH- $\lambda$
CVC4	540	424	31	696	650
Ehoh	681	418	–	691	–
Leo-III-uncoop	198	389	42	226	234
Leo-III-coop	580	<b>438</b>	43	672	668
Satallax-uncoop	–	398	43	489	507
Satallax-coop	–	432	43	602	616
Vampire	<b>731</b>	432	42	<b>719</b>	708
FOZip	399	–	–	–	–
@+FOZip	363	400	–	478	–
$\lambda$ freeZip	395	398	–	538	–
$\lambda$ Zip-base	388	408	39	420	436
$\lambda$ Zip-pragmatic	396	411	33	496	504
$\lambda$ Zip-full	171	339	34	353	361
Zip-uncoop	514	427	45	608	620
Zip-coop	631	434	<b>46</b>	715	<b>709</b>

**Fig. 5** Number of problems proved by the different provers

## 8 Discussion and Related Work

Bentkamp et al. [13] introduced four calculi for  $\lambda$ -free clausal higher-order logic organized along two axes: *intensional* versus *extensional*, and *nonpurifying* versus *purifying*. The purifying calculi flatten the clauses containing applied variables, thereby eliminating the need for superposition into variables. As we extended their work to support  $\lambda$ -expressions, we found the purification approach problematic and gave it up because it needs  $x$  to be smaller than  $x t$ , which is impossible to achieve with a term order on  $\beta\eta$ -equivalence classes. We also quickly gave up our attempt at supporting intensional higher-order logic. Extensionality is the norm for higher-order unification [32] and is mandated by the TPTP THF format [63] and in proof assistants such as HOL4, HOL Light, Isabelle/HOL, Lean, Nuprl, and PVS.

Bentkamp et al. viewed their approach as “a stepping stone towards full higher-order logic.” It already included a notion analogous to green subterms and an ARGCONG rule, which help cope with the complications occasioned by  $\beta$ -reduction.

Our clausal  $\lambda$ -superposition calculus joins the family of proof systems for higher-order logic. It is related to Andrews’s higher-order resolution [1], Huet’s constrained resolution [38], Jensen and Pietrzykowski’s  $\omega$ -resolution [40], Snyder’s higher-order  $E$ -resolution [59], Benzmüller and Kohlhase’s extensional higher-order resolution [15], Benzmüller’s higher-order unordered paramodulation and RUE resolution [14], and Bhayat and Reger’s combinatory superposition [20]. A noteworthy variant of higher-order unordered paramodulation is Steen and Benzmüller’s higher-order ordered paramodulation [61], whose order restrictions undermine refutational completeness but yield better empirical results. Other approaches are based on analytic tableaux [7, 46, 47, 55], connections [2], sequents [50], and satisfiability modulo theories (SMT) [8]. Andrews [3] and Benzmüller and Miller [16] provide excellent surveys of higher-order automation.

Combinatory superposition was developed shortly after  $\lambda$ -superposition and is closely related. It is modeled on the intensional nonpurifying calculus by Bentkamp et al. and targets extensional polymorphic clausal higher-order logic. Both combinatory and  $\lambda$ -superposition gracefully generalize the highly successful first-order superposition rules without sacrificing refutational completeness, and both are equipped with a redundancy criterion, which earlier refutationally complete higher-order calculi lack. In particular, PRUNEARG is a versatile simplification rule that could be useful in other provers. Combinatory superposition’s distinguishing feature is that it uses SKBCI combinators to represent  $\lambda$ -expressions. Combinators can be implemented more easily starting from a first-order prover;  $\beta$ -reduction amounts to demodulation. However, according to its developers [20], “Narrowing terms with combinator axioms is still explosive and results in redundant clauses. It is also never likely to be competitive with higher-order unification in finding complex unifiers.” Among the drawbacks of  $\lambda$ -superposition are the need to solve flex–flex pairs eagerly and the explosion caused by the extensionality axiom. We believe that this is a reasonable trade-off, especially for large problems with a substantial first-order component.

Our prototype Zipperposition joins the league of automatic theorem provers for higher-order logic. We list some of its rivals. TPS [4] is based on the connection method and expansion proofs. LEO [15] and LEO-II [18] implement variants of RUE resolution. Leo-III [61] is based on higher-order paramodulation. Satallax [26] implements a higher-order tableau calculus guided by a SAT solver. LEO-II, Leo-III, and Satallax integrate first-order provers as terminal procedures. agsyHOL [50] is based on a focused sequent calculus guided by narrowing. The SMT solvers CVC4 and veriT have recently been extended to higher-order logic [8]. Vampire now implements both combinatory superposition and an incomplete variant called restricted combinatory unification [19].

Half a century ago, Robinson [56] proposed to reduce higher-order logic to first-order logic via a translation. “Hammer” tools such as Sledgehammer [54], MizAR [64], HOLy-Hammer [44], and CoqHammer [31] have since popularized this approach in proof assistants. The translation must eliminate the  $\lambda$ -expressions, typically using SKBCI combinators or  $\lambda$ -lifting [52], and encode typing information [21].

## 9 Conclusion

We presented the clausal  $\lambda$ -superposition calculus, which targets a clausal fragment of extensional polymorphic higher-order logic. With the exception of a functional extensionality axiom, it gracefully generalizes standard superposition. Our prototype prover Zipperposition shows promising results on TPTP and Isabelle benchmarks. In future work, we plan to pursue five main avenues of investigation.

We first plan to *extend the calculus to support Booleans and Hilbert choice*. Booleans are notoriously explosive. We want to experiment with both axiomatizations and native support in the calculus. Native support would likely take the form of a primitive substitution rule that enumerates predicate instantiations [2], delayed clausification rules [34], and rules for reasoning about Hilbert choice.

We want to investigate techniques to *curb the explosion caused by functional extensionality*. The extensionality axiom reintroduces the search space explosion that the calculus’s order restrictions aim at avoiding. Maybe we can replace it by more restricted inference rules without compromising refutational completeness.

We will also look into approaches to *curb the explosion caused by higher-order unification*. Our calculus suffers from the need to solve flex–flex pairs. Existing procedures [40, 60, 66] enumerate redundant unifiers. This can probably be avoided to some extent. It could also be useful to investigate unification procedures that would delay imitation/projection choices via special schematic variables, inspired by Libal’s representation of regular unifiers [49].

We clearly need to *fine-tune and develop heuristics*. We expect heuristics to be a fruitful area for future research in higher-order reasoning. Proof assistants are an inexhaustible source of easy-looking benchmarks that are beyond the power of today’s provers. Whereas “hard higher-order” may remain forever out of reach, we believe that there is a substantial “easy higher-order” fragment that awaits automation.

Finally, we plan to *implement the calculus in a state-of-the-art prover*. A suitable basis for an optimized implementation of the calculus would be Ehoh, the  $\lambda$ -free clausal higher-order version of E developed by Vukmirović et al. [66].

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