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SPASS-SATT
A CDCL(LA) Solver

Martin Bromberger, Mathias Fleury, Simon Schwarz,
and Christoph Weidenbach

SIC Saarland Informatics Campus

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A CDCL(LA) Solver

Translation: fun (=SPASS) sated (=SATT)

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A CDCL(LA) Solver

Translation: fun (=SPASS) sated (=SATT)

being sick/tired of having fun…

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Quantifier-Free Linear Arithmetic

$$(x > 0 \quad \vee \quad x + y > 0) \ \wedge \ (x < 0 \quad \vee \quad x + y < 3)$$
$$\wedge \quad (y < 0) \quad \wedge \quad \neg(x > 0)$$

Quantifier-Free Linear Arithmetic

$$(x > 0 \quad \vee \quad x + y > 0) \ \wedge \ (x < 0 \quad \vee \quad x + y < 3)$$
$$\wedge \ \ (y < 0) \ \ \wedge \ \ \neg(x > 0)$$

Signature: $\Sigma_{LA} \coloneqq \{+, -, <, \leq, \geq, >, 0, 1, 2, \ldots\}$

Quantifier-Free Linear Arithmetic

$$(x > 0 \quad \vee \quad x + y > 0) \ \wedge \ (x < 0 \quad \vee \quad x + y < 3)$$
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Signature: $\Sigma_{LA} \coloneqq \{+, -, <, \leq, \geq, >, 0, 1, 2, \dots\}$

Multiplication only as syntactic sugar!

E.g.: $3 \cdot x \ \mapsto x + x + x$

Quantifier-Free Linear Arithmetic

$$(x > 0 \quad \vee \quad x + y > 0) \ \wedge \ (x < 0 \quad \vee \quad x + y < 3)$$
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Multiplication only as syntactic sugar!

E.g.: $3 \cdot x \ \mapsto x + x + x$

Goal: Quantifier-Free Linear Rational Arithmetic (QF_LRA)
⇒ rational solution, i.e., $x, y, \dots \in \mathbb{Q}$

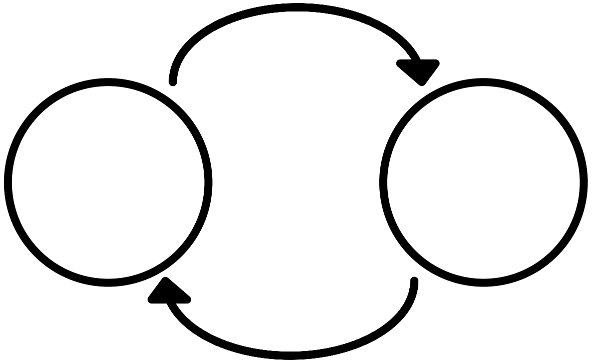
Quantifier-Free Linear Integer Arithmetic (QF_LIA)
⇒ integer solution, i.e., $x, y, \dots \in \mathbb{Z}$

CDCL(T)

$$(x > 0 \quad \vee \quad x + y > 0) \;\wedge\; (x < 0 \quad \vee \quad x + y < 3)$$
$$\wedge \;\; (y < 0) \;\; \wedge \;\; \neg(x > 0)$$

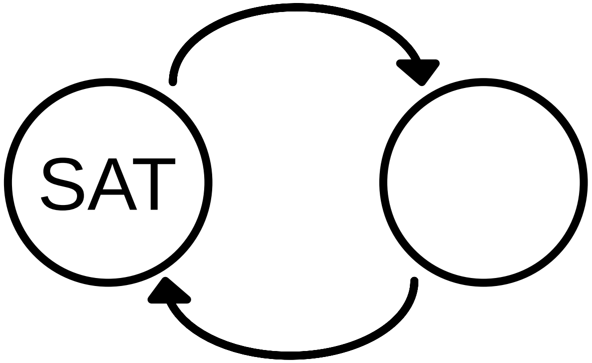
CDCL(T)

$$(x > 0 \quad \vee \quad x + y > 0) \ \wedge \ (x < 0 \quad \vee \quad x + y < 3)$$
$$\wedge \quad (y < 0) \ \wedge \ \neg(x > 0)$$



CDCL(T)

$$(x > 0 \quad \vee \quad x + y > 0) \wedge (x < 0 \quad \vee \quad x + y < 3)$$
$$\wedge \quad (y < 0) \quad \wedge \quad \neg(x > 0)$$



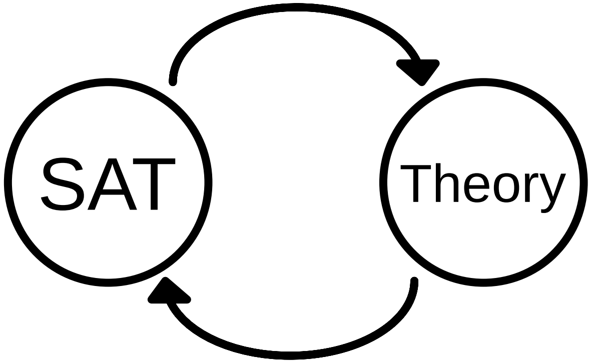
CDCL solver:

CDCL = conflict-driven clause-learning

Decision procedure for propositional CNF formulas

CDCL(T)

$$(x > 0 \quad \vee \quad x + y > 0) \wedge (x < 0 \quad \vee \quad x + y < 3)$$
$$\wedge \quad (y < 0) \quad \wedge \quad \neg(x > 0)$$



CDCL solver:

CDCL = conflict-driven clause-learning

Decision procedure for propositional CNF formulas

Theory solver:

Decision procedure for conjunctions of theory atoms

e.g. Simplex for QF_LRA & Branch-and-Bound for QF_LIA

CDCL(LA)

$$(x > 0 \quad \vee \quad x + y > 0) \ \wedge \ (x < 0 \quad \vee \quad x + y < 3)$$
$$\wedge \quad (y < 0) \quad \wedge \quad \neg(x > 0)$$

CDCL(LA)

$$(x > 0 \quad \vee \quad x + y > 0) \ \wedge \ (x < 0 \quad \vee \quad x + y < 3)$$
$$\wedge \quad (y < 0) \quad \wedge \quad \neg(x > 0)$$

CDCL(LA)

$$(x > 0 \quad \lor \quad x + y > 0) \;\land\; (x < 0 \quad \lor \quad x + y < 3)$$
$$\land \;\; (y < 0) \;\; \land \;\; \neg(x > 0)$$

$$\boxed{A \Leftrightarrow x > 0;}$$

CDCL(LA)

$$(\boxed{A} \vee x + y > 0) \wedge (x < 0 \vee x + y < 3)$$
$$\wedge (y < 0) \wedge \neg(\boxed{A})$$

$$\boxed{A \Leftrightarrow x > 0;}$$

CDCL(LA)

$$(\boxed{A} \quad \vee \quad x + y > 0) \wedge (x < 0 \quad \vee \quad x + y < 3)$$

$$\wedge \ (y < 0) \ \wedge \ \neg(\boxed{A})$$

$\boxed{A \Leftrightarrow x > 0;}$

$\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$

$\boxed{D \Leftrightarrow x + y > 4;}$

$\boxed{E \Leftrightarrow y < 0;}$

CDCL(LA)

$$(\boxed{A} \vee \boxed{B}) \wedge (\boxed{C} \vee \boxed{D})$$
$$\wedge (\boxed{E}) \wedge \neg(\boxed{A})$$

$$\boxed{A \Leftrightarrow x > 0;} \quad \boxed{B \Leftrightarrow x + y > 0;}$$

$$\boxed{C \Leftrightarrow x < 0;} \quad \boxed{D \Leftrightarrow x + y > 4;}$$

$$\boxed{E \Leftrightarrow y < 0;}$$

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CDCL(LA)

$$(\boxed{A} \lor \boxed{B}) \land (\boxed{C} \lor \boxed{D}) \land \boxed{E} \land \neg \boxed{A}$$

Unit Propagation

Model:

$\boxed{A \Leftrightarrow x > 0;}$ $\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$ $\boxed{D \Leftrightarrow x + y > 4;}$

$\boxed{E \Leftrightarrow y < 0;}$

CDCL(LA)

$$(\boxed{A} \lor \boxed{B}) \land (\boxed{C} \lor \boxed{D}) \land \overset{\Downarrow}{\boxed{E}} \land \neg\boxed{A}$$

Unit Propagation

Model:

$\boxed{A \Leftrightarrow x > 0;}$

$\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$

$\boxed{D \Leftrightarrow x + y > 4;}$

$\boxed{E \Leftrightarrow y < 0;}$

CDCL(LA)

$$(\boxed{A} \lor \boxed{B}) \land (\boxed{C} \lor \boxed{D}) \land \underset{\mathrm{T}}{\overset{\Downarrow}{\boxed{E}}} \land \neg \boxed{A}$$

Unit Propagation

Model: $\boxed{E}$

$\boxed{A \Leftrightarrow x > 0;}$

$\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$

$\boxed{D \Leftrightarrow x + y > 4;}$

$\boxed{E \Leftrightarrow y < 0;}$

CDCL(LA)

$$(\boxed{A} \lor \boxed{B}) \land (\boxed{C} \lor \boxed{D}) \land \underset{\top}{\boxed{E}} \land \neg \boxed{A}$$

Unit Propagation

Model: $\boxed{E}$

$\boxed{A \Leftrightarrow x > 0;}$ $\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$ $\boxed{D \Leftrightarrow x + y > 4;}$

$\boxed{E \Leftrightarrow y < 0;}$

CDCL(LA)

$$(\boxed{A} \vee \boxed{B}) \wedge (\boxed{C} \vee \boxed{D}) \wedge \underset{\top}{\boxed{E}} \wedge \neg \overset{\Downarrow}{\boxed{A}}$$

Unit Propagation

Model: $\boxed{E}$

$\boxed{A \Leftrightarrow x > 0;}$

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CDCL(LA)

$$(\underset{\bot}{\boxed{A}} \vee \boxed{B}) \wedge (\boxed{C} \vee \boxed{D}) \wedge \underset{\top}{\boxed{E}} \wedge \neg \underset{\top}{\overset{\Downarrow}{\boxed{A}}}$$

Unit Propagation

Model: $\boxed{E}$ $\boxed{\neg A}$

$\boxed{A \Leftrightarrow x > 0;}$

$\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$

$\boxed{D \Leftrightarrow x + y > 4;}$

$\boxed{E \Leftrightarrow y < 0;}$

CDCL(LA)

$$(\boxed{A} \lor \boxed{B}) \land (\boxed{C} \lor \boxed{D}) \land \boxed{E} \land \neg \boxed{A}$$

$$\underset{\bot}{} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \top \qquad \top$$

Unit Propagation

Model: $\boxed{E}$ $\boxed{\neg A}$

$\boxed{A \Leftrightarrow x > 0;}$ $\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$ $\boxed{D \Leftrightarrow x + y > 4;}$

$\boxed{E \Leftrightarrow y < 0;}$

CDCL(LA)

$$(\underset{\perp}{\boxed{A}} \lor \overset{\Downarrow}{\boxed{B}}) \land (\boxed{C} \lor \boxed{D}) \land \underset{\top}{\boxed{E}} \land \neg \underset{\top}{\boxed{A}}$$

Unit Propagation

Model: $\boxed{E}$ $\boxed{\neg A}$

$\boxed{A \Leftrightarrow x > 0;}$

$\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$

$\boxed{D \Leftrightarrow x + y > 4;}$

$\boxed{E \Leftrightarrow y < 0;}$

CDCL(LA)

$$(\underset{\bot}{\boxed{A}} \vee \overset{\Downarrow}{\underset{\top}{\boxed{B}}}) \wedge (\boxed{C} \vee \boxed{D}) \wedge \underset{\top}{\boxed{E}} \wedge \neg \underset{\top}{\boxed{A}}$$

Unit Propagation

Model: $\boxed{E}$ $\boxed{\neg A}$ $\boxed{B}$

$\boxed{A \Leftrightarrow x > 0;}$

$\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$

$\boxed{D \Leftrightarrow x + y > 4;}$

$\boxed{E \Leftrightarrow y < 0;}$

CDCL(LA)

$$(\underset{\bot}{\boxed{A}} \lor \underset{\top}{\boxed{B}}) \land (\boxed{C} \lor \boxed{D}) \land \underset{\top}{\boxed{E}} \land \neg \underset{\top}{\boxed{A}}$$

Unit Propagation

Model: $\boxed{E}$ $\boxed{\neg A}$ $\boxed{B}$

$\boxed{A \Leftrightarrow x > 0;}$ $\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$ $\boxed{D \Leftrightarrow x + y > 4;}$

$\boxed{E \Leftrightarrow y < 0;}$

CDCL(LA)

$$(\underset{\bot}{\boxed{A}} \lor \underset{\top}{\boxed{B}}) \land (\boxed{C} \lor \boxed{D}) \land \underset{\top}{\boxed{E}} \land \neg \underset{\top}{\boxed{A}}$$

Decision

Model: $\boxed{E}$ $\boxed{\neg A}$ $\boxed{B}$

$\boxed{A \Leftrightarrow x > 0;}$ $\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$ $\boxed{D \Leftrightarrow x + y > 4;}$

$\boxed{E \Leftrightarrow y < 0;}$

CDCL(LA)

$$(\underset{\bot}{\boxed{A}} \lor \underset{\top}{\boxed{B}}) \land (\underset{\top}{\boxed{C}} \lor \boxed{D}) \land \underset{\top}{\boxed{E}} \land \neg \underset{\top}{\boxed{A}}$$

Decision

Model: $\boxed{E}$ $\boxed{\neg A}$ $\boxed{B}$ $\boxed{C^{\dagger}}$

$\boxed{A \Leftrightarrow x > 0;}$

$\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$

$\boxed{D \Leftrightarrow x + y > 4;}$

$\boxed{E \Leftrightarrow y < 0;}$

CDCL(LA)

$$\begin{array}{ccccccccccc} (\boxed{A} & \vee & \boxed{B}) & \wedge & (\boxed{C} & \vee & \boxed{D}) & \wedge & \boxed{E} & \wedge & \neg\boxed{A} \\ \bot & & \top & & \top & & & & \top & & \top \end{array}$$

Theory Satisfiable?

Model: $\boxed{E}$ $\boxed{\neg A}$ $\boxed{B}$ $\boxed{C^\dagger}$

$\boxed{A \Leftrightarrow x > 0;}$ $\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$ $\boxed{D \Leftrightarrow x + y > 4;}$

$\boxed{E \Leftrightarrow y < 0;}$

CDCL(LA)

$$\begin{array}{ccccccccccc} (\boxed{A} & \vee & \boxed{B}) & \wedge & (\boxed{C} & \vee & \boxed{D}) & \wedge & \boxed{E} & \wedge & \neg\boxed{A} \\ \bot & & \top & & \top & & & & \top & & \top \end{array}$$

Theory Satisfiable? No!

Model: $\boxed{E}$ $\boxed{\neg A}$ $\boxed{B}$ $\boxed{C^{\dagger}}$

$\boxed{A \Leftrightarrow x > 0;}$ $\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$ $\boxed{D \Leftrightarrow x + y > 4;}$

$\boxed{E \Leftrightarrow y < 0;}$

CDCL(LA)

$$(\underset{\bot}{\boxed{A}} \lor \underset{\top}{\boxed{B}}) \land (\underset{\top}{\boxed{C}} \lor \boxed{D}) \land \underset{\top}{\boxed{E}} \land \neg \underset{\top}{\boxed{A}}$$

Theory Satisfiable? No!

Model: $\boxed{E}$ $\boxed{\neg A}$ $\boxed{B}$ $\boxed{C^{\dagger}}$

$\boxed{A \Leftrightarrow x > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$

$\boxed{E \Leftrightarrow y < 0;}$

$\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{D \Leftrightarrow x + y > 4;}$

$$\boxed{\begin{array}{c} \boxed{E \Leftrightarrow y < 0;} \\ \boxed{\neg A \Leftrightarrow x \leq 0;} \\ \boxed{B \Leftrightarrow x + y > 0;} \end{array}}$$

CDCL(LA)

$$(\underset{\bot}{\boxed{A}} \lor \underset{\top}{\boxed{B}}) \land (\underset{\top}{\boxed{C}} \lor \boxed{D}) \land \underset{\top}{\boxed{E}} \land \neg \underset{\top}{\boxed{A}}$$

Conflict Analysis:

Model: $\boxed{E}$ $\boxed{\neg A}$ $\boxed{B}$ $\boxed{C^{\dagger}}$

$\boxed{A \Leftrightarrow x > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$

$\boxed{E \Leftrightarrow y < 0;}$

$\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{D \Leftrightarrow x + y > 4;}$

$\boxed{E \Leftrightarrow y < 0;}$

$\boxed{\neg A \Leftrightarrow x \leq 0;}$

$\boxed{B \Leftrightarrow x + y > 0;}$

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CDCL(LA)

$$(\underset{\bot}{\boxed{A}} \vee \underset{\top}{\boxed{B}}) \wedge (\underset{\top}{\boxed{C}} \vee \boxed{D}) \wedge \underset{\top}{\boxed{E}} \wedge \neg \underset{\top}{\boxed{A}}$$

Conflict Analysis:

Model: $\boxed{E}$ $\boxed{\neg A}$ $\boxed{B}$ $\boxed{C^{\dagger}}$

$$(\neg \boxed{E} \wedge \boxed{A} \wedge \neg \boxed{B})$$

$\boxed{A \Leftrightarrow x > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$

$\boxed{E \Leftrightarrow y < 0;}$

$\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{D \Leftrightarrow x + y > 4;}$

$\boxed{E \Leftrightarrow y < 0;}$

$\boxed{\neg A \Leftrightarrow x \leq 0;}$

$\boxed{B \Leftrightarrow x + y > 0;}$

CDCL(LA)

$$(\underset{\bot}{\boxed{A}} \lor \underset{\top}{\boxed{B}}) \land (\underset{\top}{\boxed{C}} \lor \boxed{D}) \land \underset{\top}{\boxed{E}} \land \neg \underset{\top}{\boxed{A}}$$

Conflict Analysis:

Model: $\boxed{E}$ $\boxed{\neg A}$ $\boxed{B}$ $\boxed{C^{\dagger}}$

$$(\neg \underset{\bot}{\boxed{E}} \land \underset{\bot}{\boxed{A}} \land \neg \underset{\bot}{\boxed{B}})$$

$\boxed{A \Leftrightarrow x > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$

$\boxed{E \Leftrightarrow y < 0;}$

$\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{D \Leftrightarrow x + y > 4;}$

$\boxed{E \Leftrightarrow y < 0;}$

$\boxed{\neg A \Leftrightarrow x \leq 0;}$

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CDCL(LA)

$$(\underset{\bot}{\boxed{A}} \lor \underset{\top}{\boxed{B}}) \land (\underset{\top}{\boxed{C}} \lor \boxed{D}) \land \underset{\top}{\boxed{E}} \land \neg \underset{\top}{\boxed{A}}$$

Conflict Analysis: UNSAT!

Model: $\boxed{E}$ $\boxed{\neg A}$ $\boxed{B}$ $\boxed{C^{\dagger}}$

$$(\neg \underset{\bot}{\boxed{E}} \land \underset{\bot}{\boxed{A}} \land \neg \underset{\bot}{\boxed{B}})$$

$\boxed{A \Leftrightarrow x > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$

$\boxed{E \Leftrightarrow y < 0;}$

$\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{D \Leftrightarrow x + y > 4;}$

$\boxed{E \Leftrightarrow y < 0;}$

$\boxed{\neg A \Leftrightarrow x \leq 0;}$

$\boxed{B \Leftrightarrow x + y > 0;}$

SMT-COMP 2018

QF_LIA (Main Track)

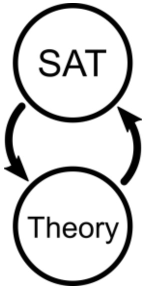
QF_LIA = quantifier-free linear integer arithmetic
Benchmarks: 6947
Time limit: 1200s

Solver	Solved Score	CPU time Score	Solved
SPASS-SATT	6587.626	72.048	6744
Ctrl-Ergo	6221.467	156.086	6259
MathSAT^n	6135.114	164.626	6528
SMTInterpol	5915.623	204.123	6286
CVC4	5891.019	194.986	6357
Yices 2.6.0	5867.976	209.452	6232
z3-4.7.1^n	5733.374	224.539	6195
SMTRAT-Rat	4049.914	515.394	3112
veriT	3155.162	295.434	2734

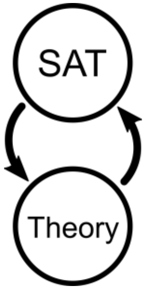
QF_LRA (Main Track)

QF_LRA = quantifier-free linear rational arithmetic
Benchmarks: 1649
Time limit: 1200s

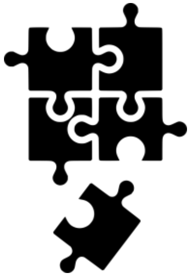
Solver	Solved Score	CPU time Score	Solved
CVC4	1586.833	69.006	1566
SPASS-SATT	1586.396	64.292	1590
Yices 2.6.0	1583.186	63.901	1567
veriT	1568.212	79.840	1527
SMTInterpol	1548.476	102.257	1521
MathSAT^n	1536.458	107.673	1461
z3-4.7.1^n	1527.249	113.154	1435
opensmt2	1498.663	131.674	1329
Ctrl-Ergo	1450.082	172.097	1354
SMTRAT-Rat	1297.891	275.918	984
SMTRAT-MCSAT	1090.526	409.015	711



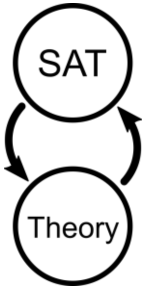
SAT and theory interaction:



SAT and theory interaction:



Theory solver extensions:



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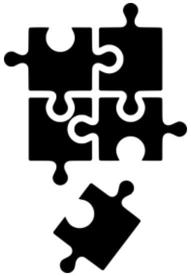

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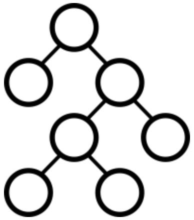
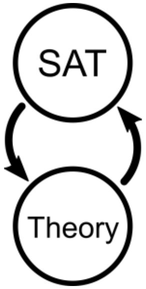


Data-structure improvements:



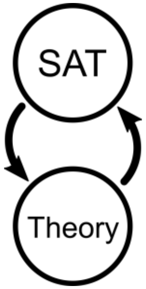
SAT and theory interaction:

Theory solver extensions:

Data-structure improvements:

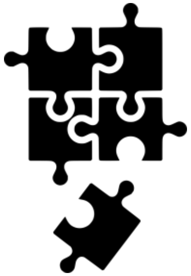
Preprocessing:

[…] invented by our team […] invented & published by someone else […] never published but implemented



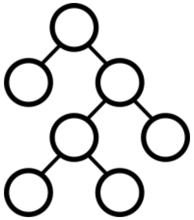
SAT and theory interaction:

- weakened early pruning [Sebastiani07]
- unate propagations and bound refinements [Dutertre06]
- decision recommendations [Yices]



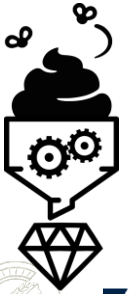
Theory solver extensions:

- unit cube test [Bromberger16]
- bounding transformation [Bromberger18]
- simple rounding and bound propagation [Schrijver86]



Data-structure improvements:

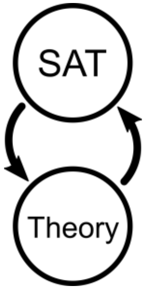
- priority queue for pivot selection [pretty much everyone]
- integer coefficients instead of rational coefficients [veriT]
- backup instead of recalculation [pretty much everyone]



Preprocessing:

- if-then-else (reconstruction, lifting, simplification, bounding) [CVC4]
- pseudo-Boolean inequalities [CVC4]
- small CNF transformation [Weidenbach01]

[…] invented by our team [...] invented & published by someone else […] never published but implemented



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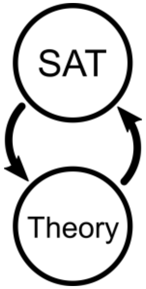
- small CNF transformation [Weidenbach01]

SAT and Theory Interaction



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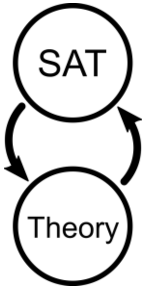

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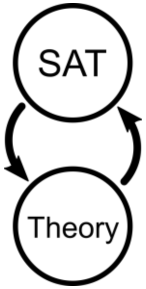

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(Weakened) early pruning [Sebastiani07]

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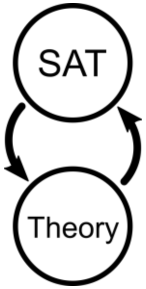

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- unate propagations and bound refinements [Dutertre06]

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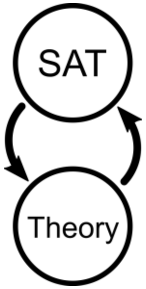

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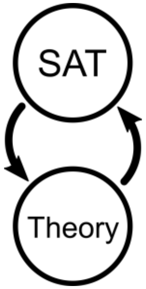

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Early Pruning

$$\begin{array}{ccccccccccc} (\boxed{A} & \lor & \boxed{B}) & \land & (\boxed{C} & \lor & \boxed{D}) & \land & \boxed{E} & \land & \neg\boxed{A} \\ \bot & & \top & & \top & & & & \top & & \top \end{array}$$

Theory Satisfiable? No!

Model: $\boxed{E}$ $\boxed{\neg A}$ $\boxed{B}$ $\boxed{C^{\dagger}}$

$\boxed{A \Leftrightarrow x > 0;}$

$\boxed{C \Leftrightarrow x < 0;}$

$\boxed{E \Leftrightarrow y < 0;}$

$\boxed{B \Leftrightarrow x + y > 0;}$

$\boxed{D \Leftrightarrow x + y > 4;}$

$$\boxed{\begin{array}{c} \boxed{E \Leftrightarrow y < 0;} \\ \boxed{\neg A \Leftrightarrow x \leq 0;} \\ \boxed{B \Leftrightarrow x + y > 0;} \end{array}}$$

Early Pruning

$$(\underset{\bot}{\boxed{A}} \lor \underset{\top}{\boxed{B}}) \land (\boxed{C} \lor \boxed{D}) \land \underset{\top}{\boxed{E}} \land \neg \underset{\top}{\boxed{A}}$$

Check for theory satisfiability before each decision!

$$\boxed{A \Leftrightarrow x > 0;} \qquad \boxed{B \Leftrightarrow x + y > 0;}$$

$$\boxed{C \Leftrightarrow x < 0;} \qquad \boxed{D \Leftrightarrow x + y > 4;}$$

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Full theory check is too expensive? (NP for QF_LIA)

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Weakened Early Pruning

$$(\underset{\bot}{\boxed{A}} \vee \underset{\top}{\boxed{B}}) \wedge (\boxed{C} \vee \boxed{D}) \wedge \underset{\top}{\boxed{E}} \wedge \neg \underset{\top}{\boxed{A}}$$

Check for theory satisfiability before each decision!

Full theory check is too expensive? (NP for QF_LIA)

Do a weaker check! (Check only for rational solutions)

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$$\boxed{C \Leftrightarrow x < 0;} \quad \boxed{D \Leftrightarrow x + y > 4;}$$

$$\boxed{E \Leftrightarrow y < 0;}$$

Decision Recommendations

How to select phase of decision literal? $\boxed{C^\dagger}$ or $\boxed{\neg C^\dagger}$

$\boxed{A \Leftrightarrow x \geq 0;}$

$\boxed{B \Leftrightarrow y \geq x + 1;}$

$\boxed{C \Leftrightarrow y \geq 5;}$

Model: $\boxed{A}$ $\boxed{B}$

Decision Recommendations

How to select phase of decision literal? $\boxed{C^{\dagger}}$ or $\boxed{\neg C^{\dagger}}$

Use rational assignment as heuristic

(Assignment is side effect of failed weakened early pruning)

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Assignment: $x = 0, y = 1$

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Goal: assignment should stay solution for model

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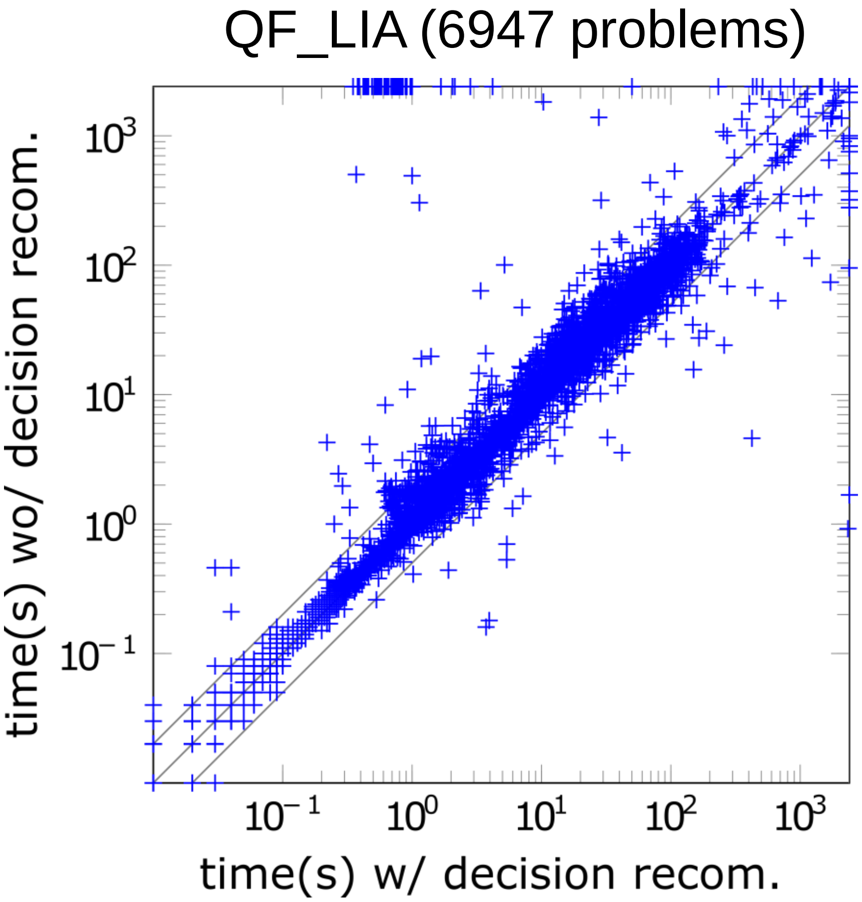

additional instances: 129

twice as fast/slow: 389/58

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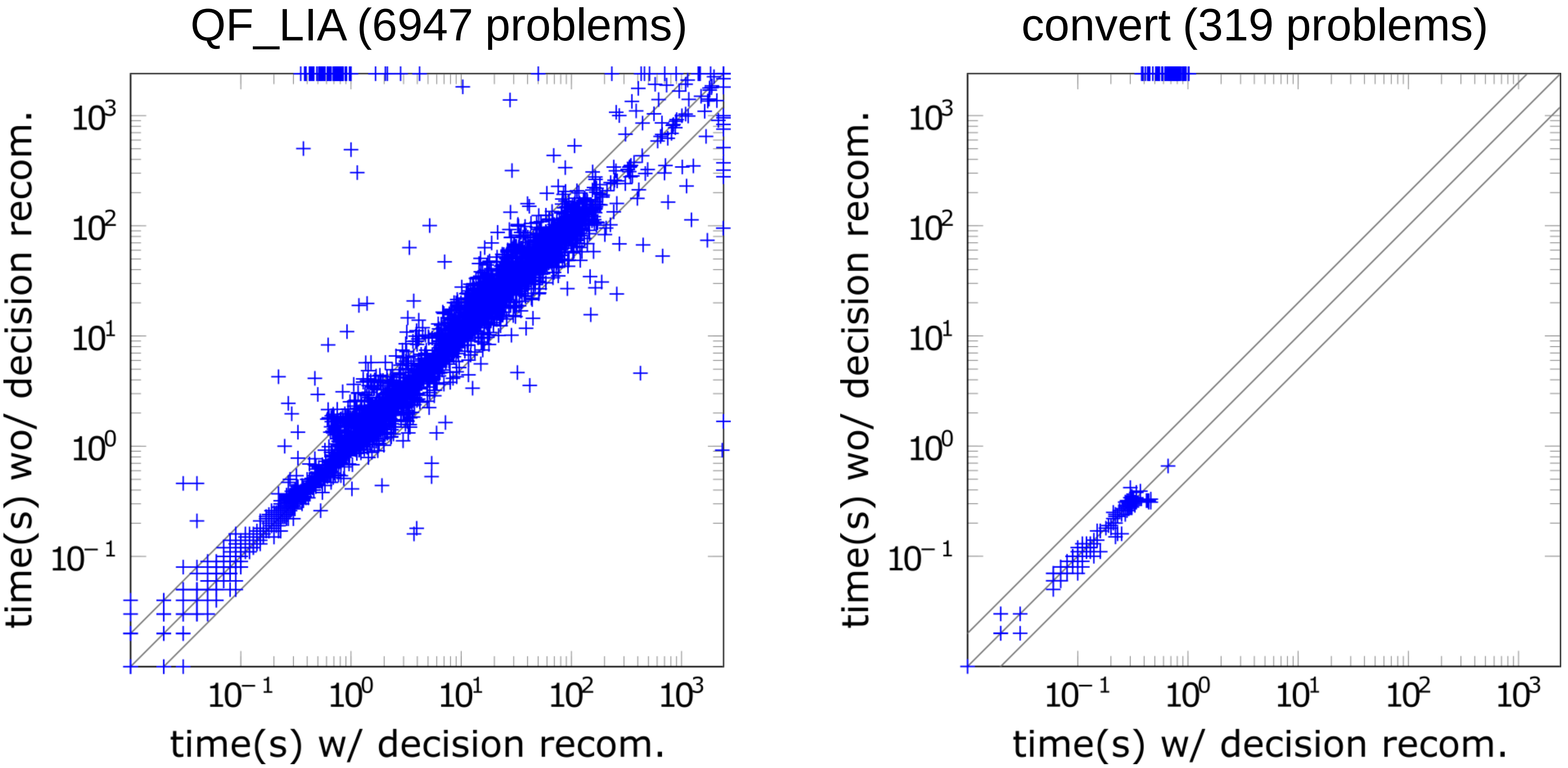

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Theory Solver

Input: $\{a_i^T x \leq b_i \mid i = 1, \dots, m\}$

Goal: QF_LRA: $x_1, \dots, x_n \in \mathbb{Q}$ or QF_LIA: $x_1, \dots, x_n \in \mathbb{Z}$

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Example:

$$\boxed{\begin{array}{ll} 2x_2 \leq 5x_1, & 3x_2 \geq 4x_1, \\ 2x_2 \leq -5x_1 + 15, & 2x_2 \geq -3x_1 + 4, \end{array}}$$

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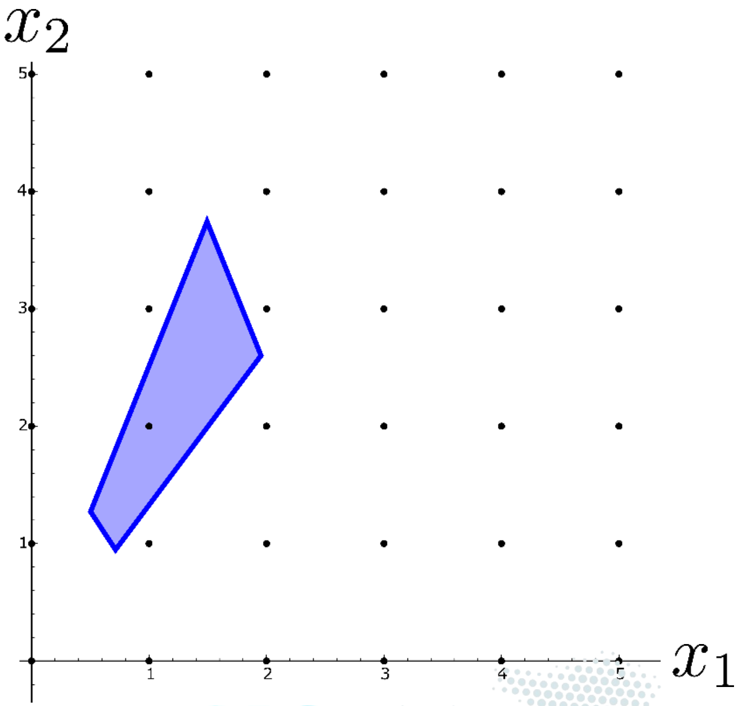

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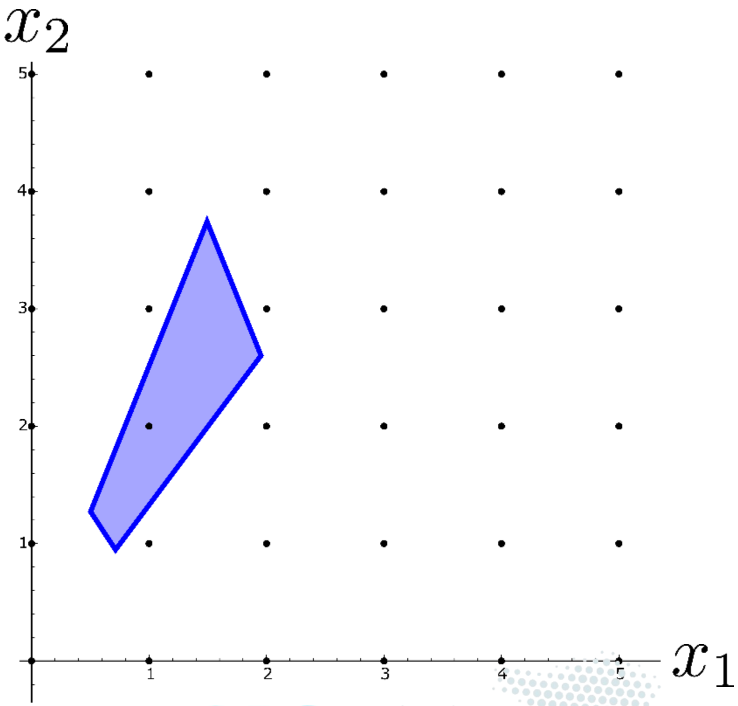

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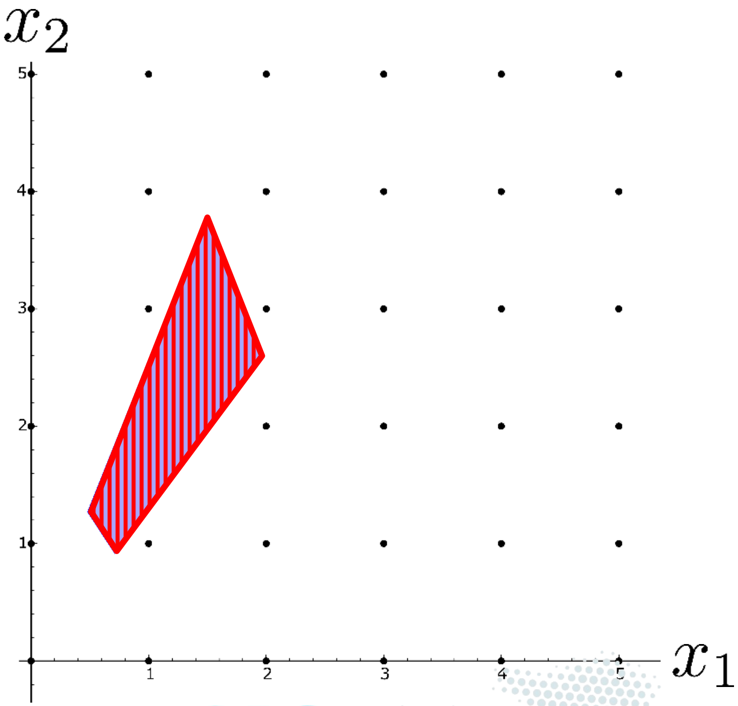

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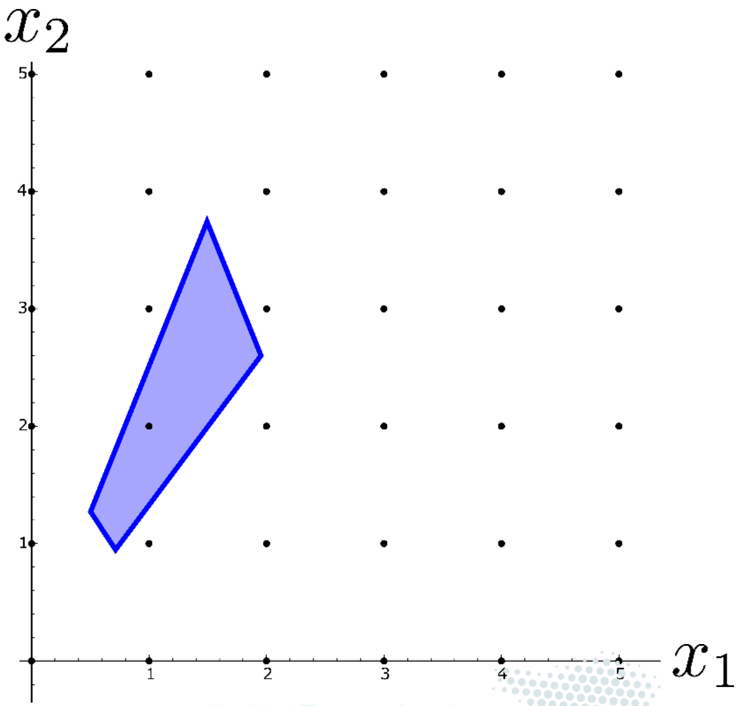

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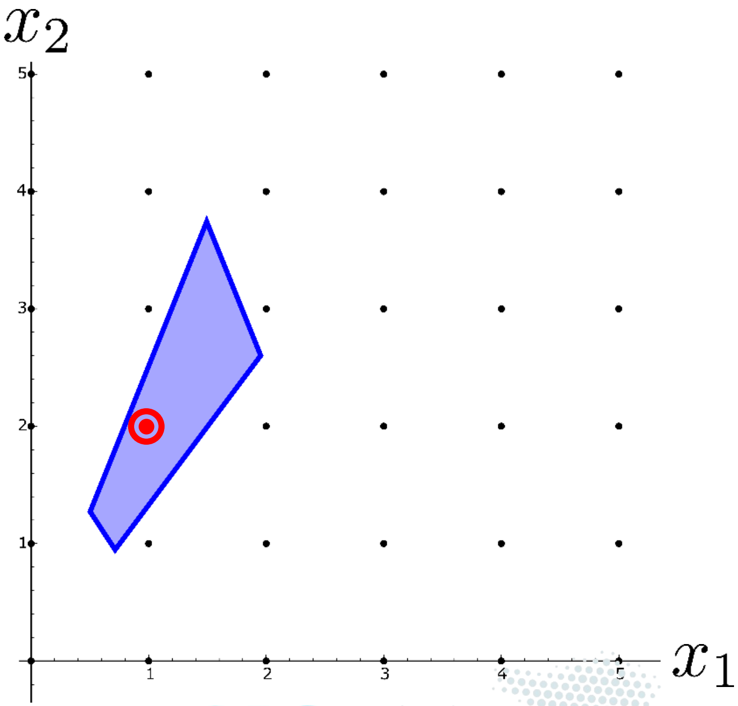

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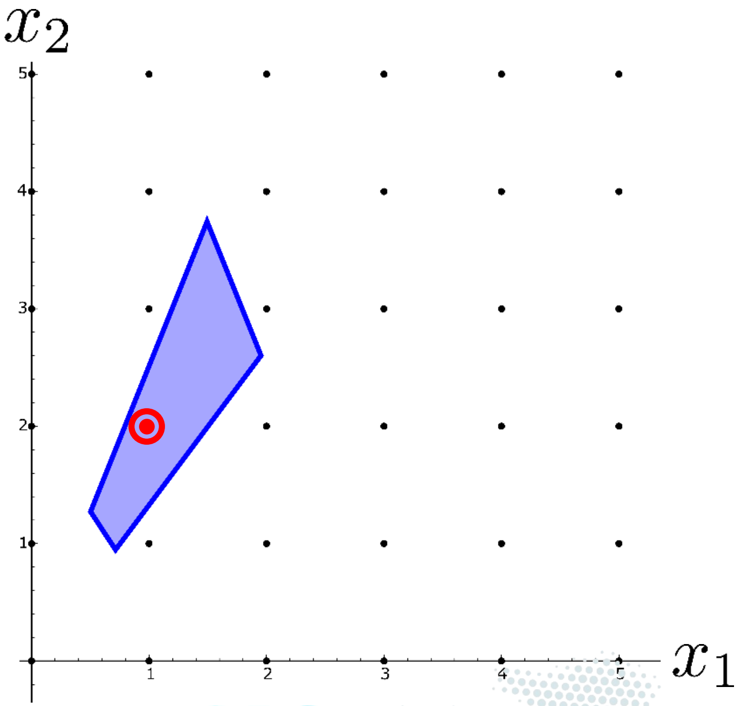

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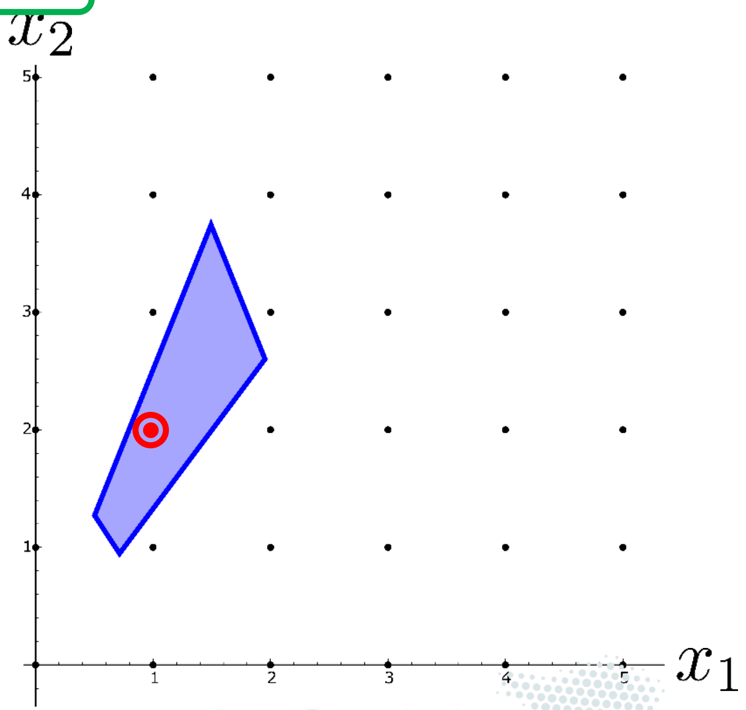
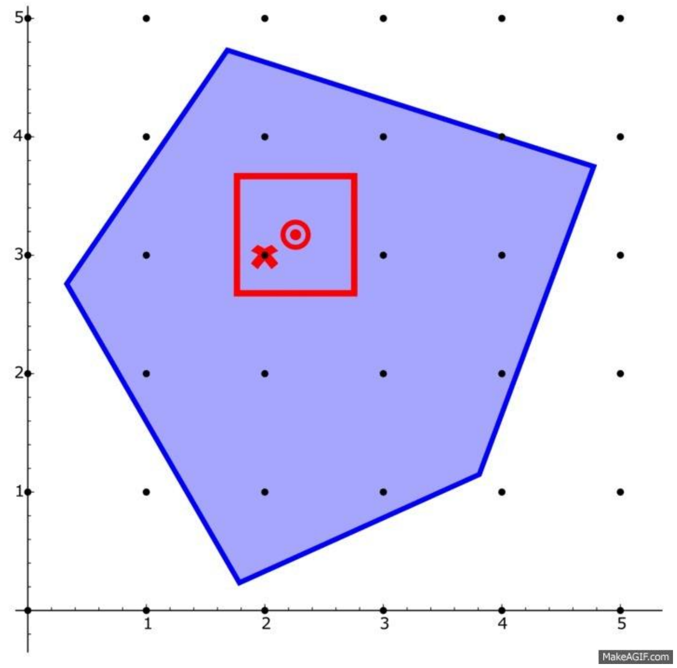


Theory Solver Extensions

Unit Cube Test

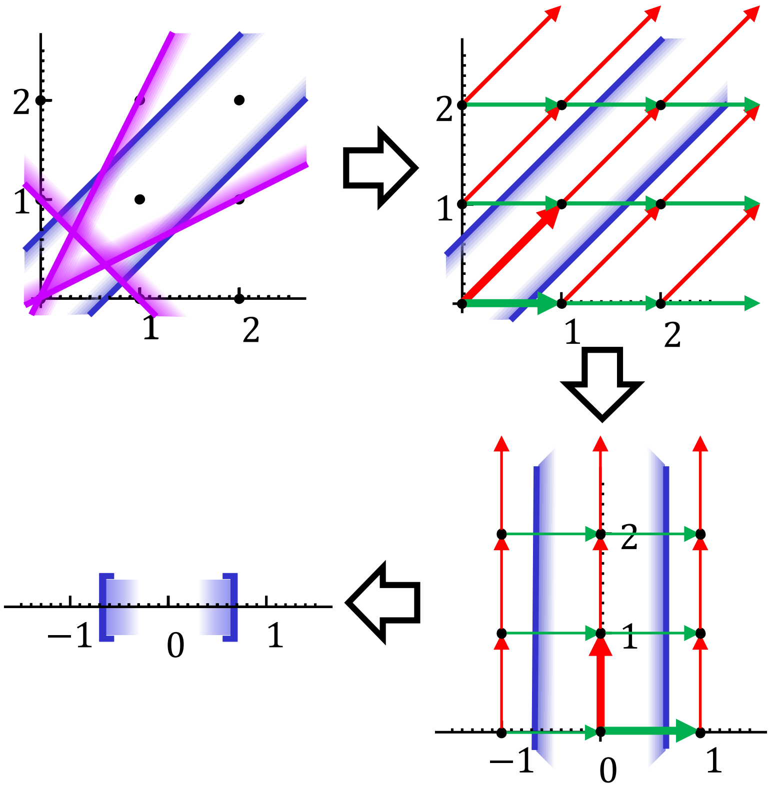
(IJCAR 2016)



for absolutely unbounded problems

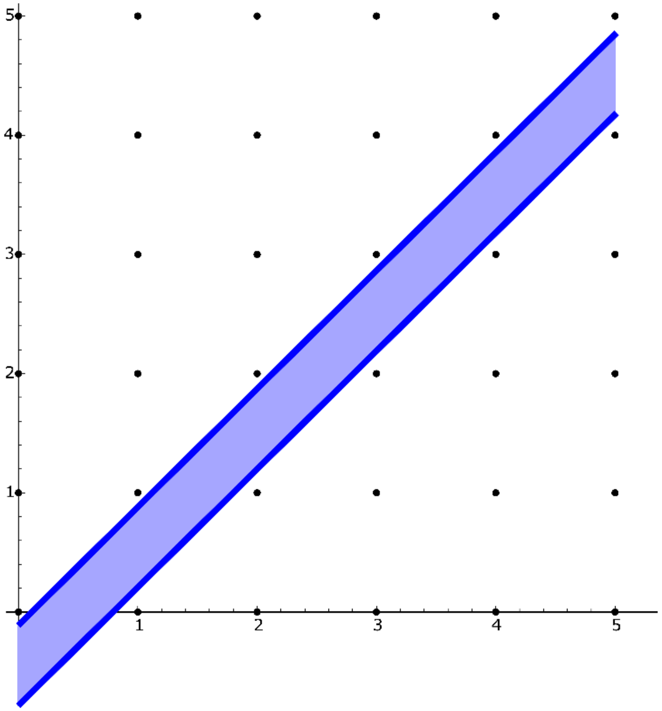
Bounding Transformation

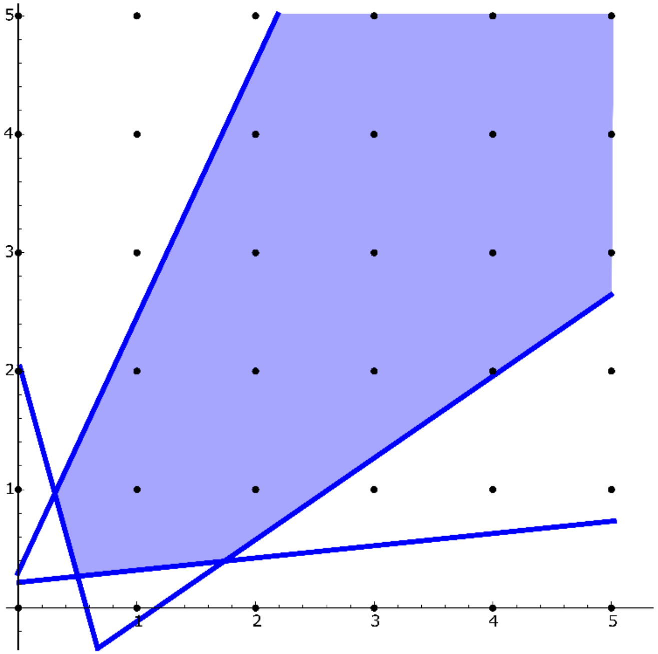
(IJCAR 2018)



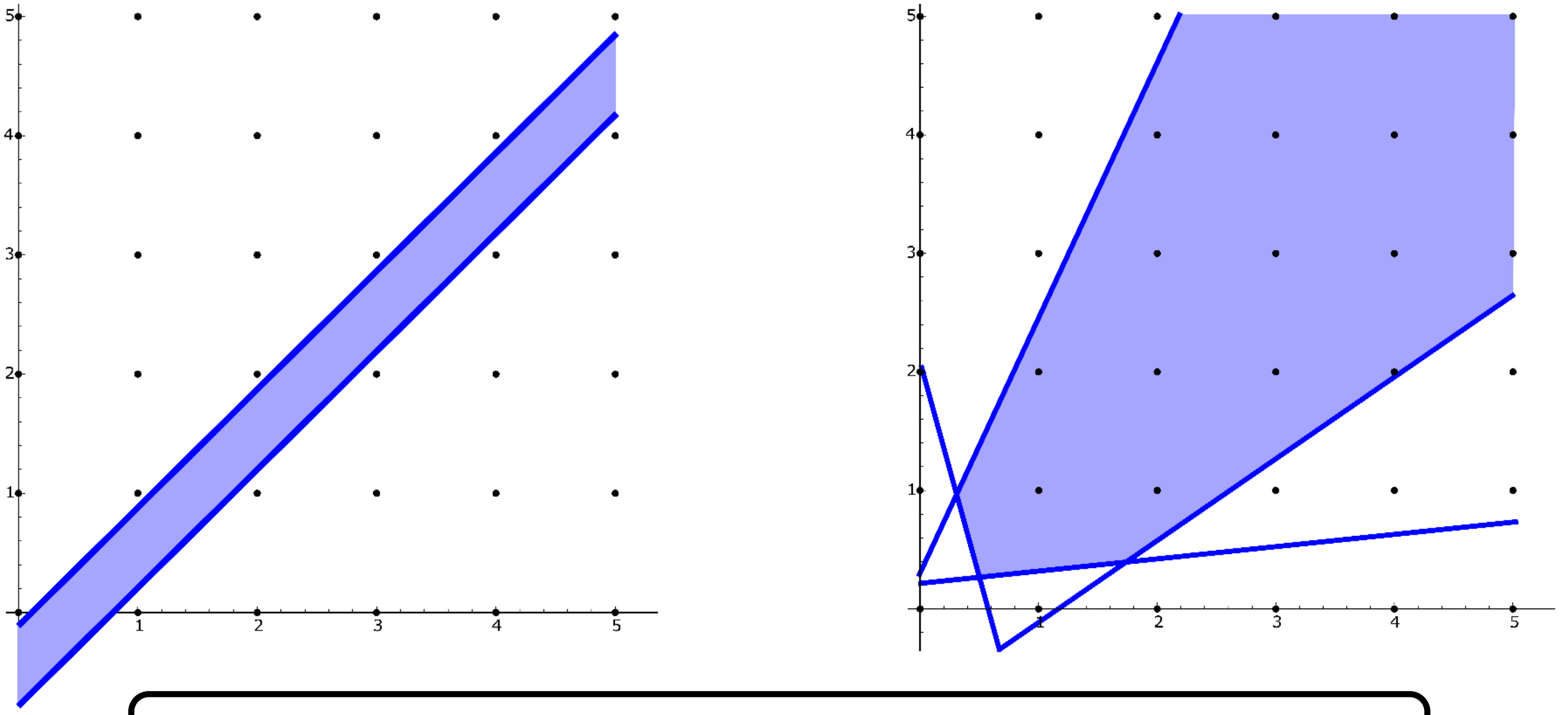
for partially unbounded problems

Unbounded Problems



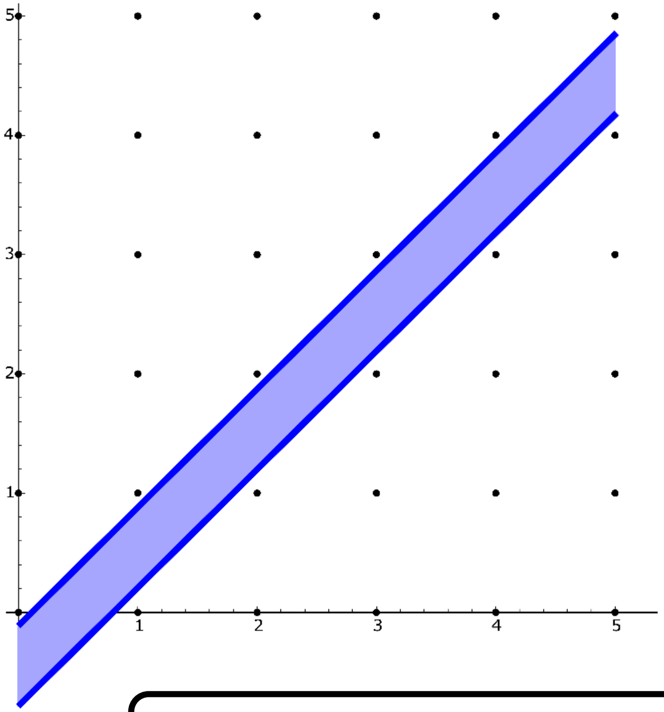


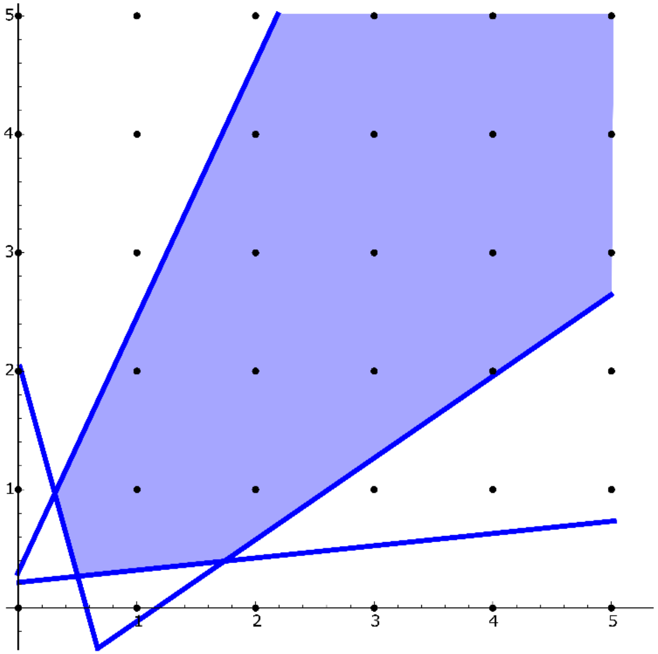
Unbounded Problems



Requirement: unbounded direction

Unbounded Problems





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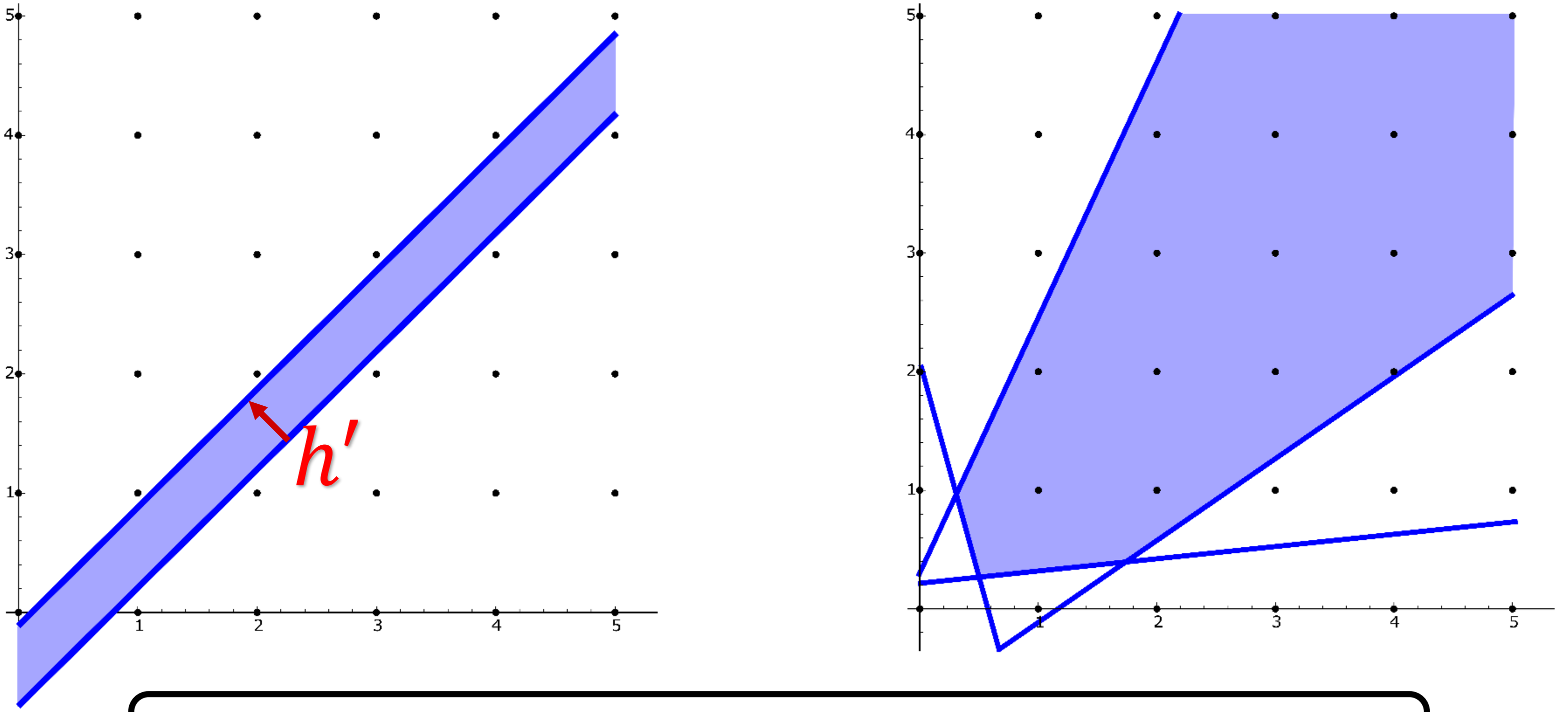
$h \in \mathbb{Q}^n$ is bounded iff

$$\exists l, u \in \mathbb{Z}.\ \forall x \in \mathbb{Q}^n. \{a_i^T x \le b_i \mid i = 1, \dots, m\} \to \boxed{l \le} h^T x \boxed{\le u}$$

lower bound upper bound

Unbounded Problems



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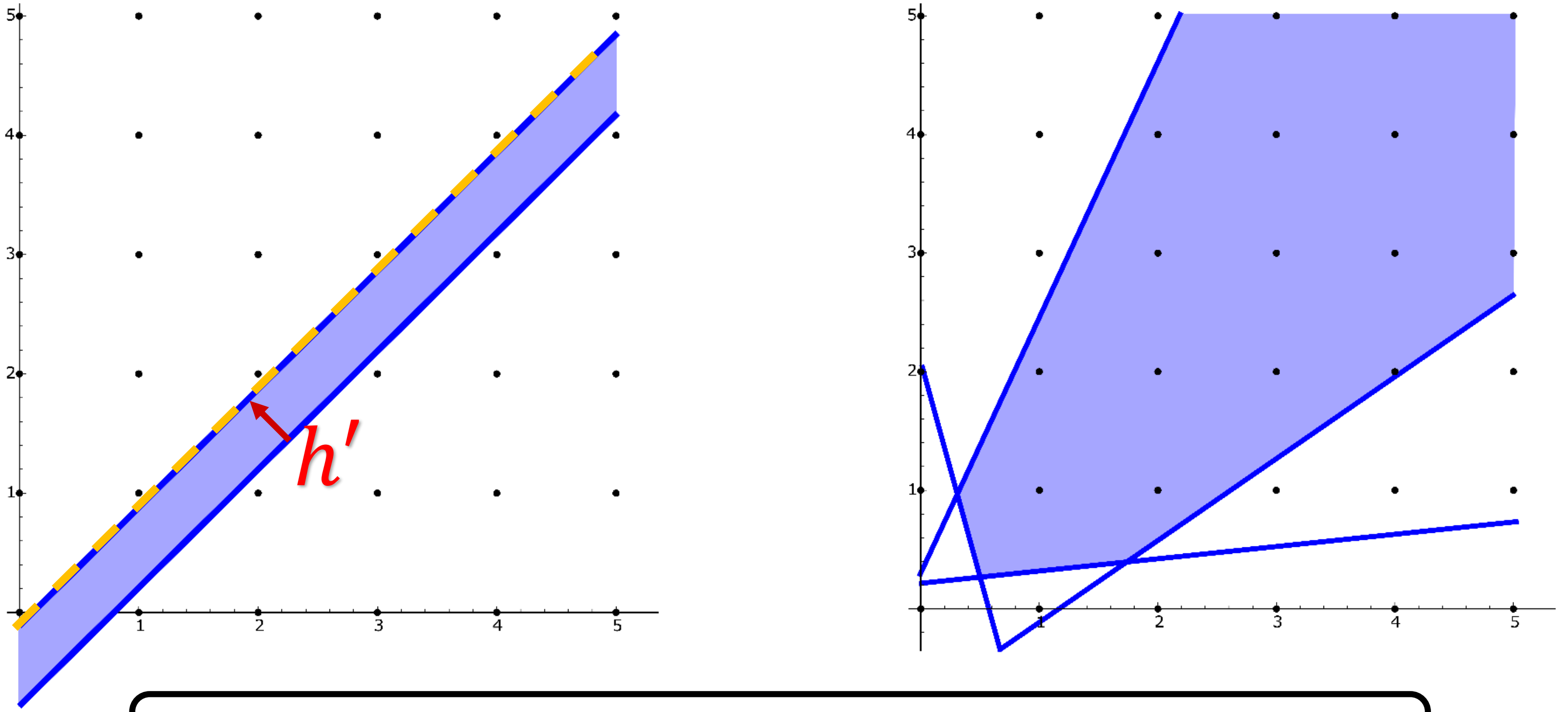
Unbounded Problems



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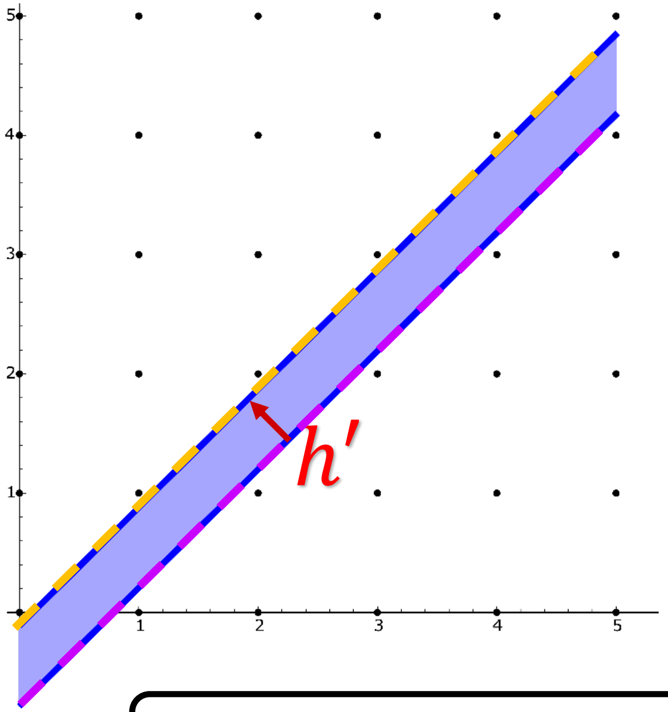

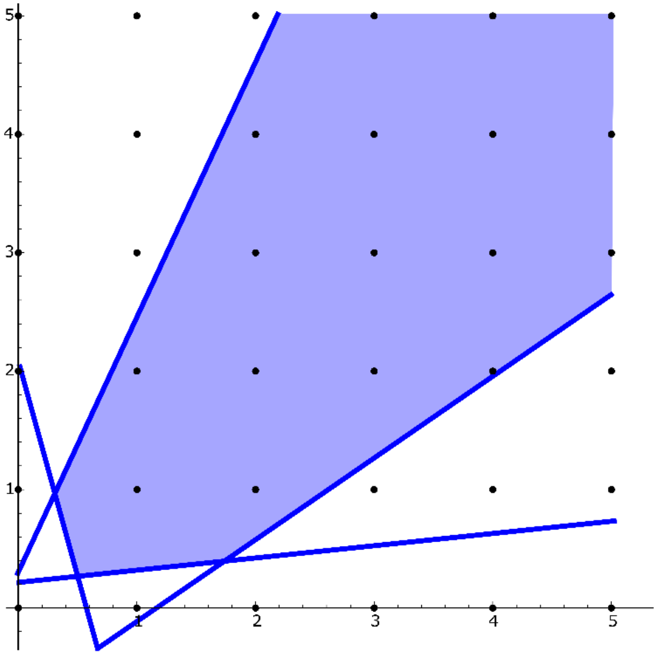

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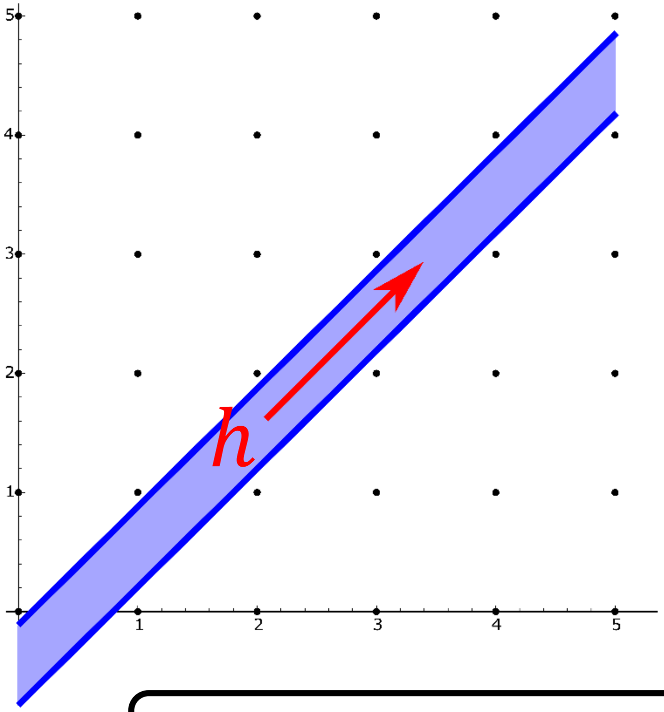

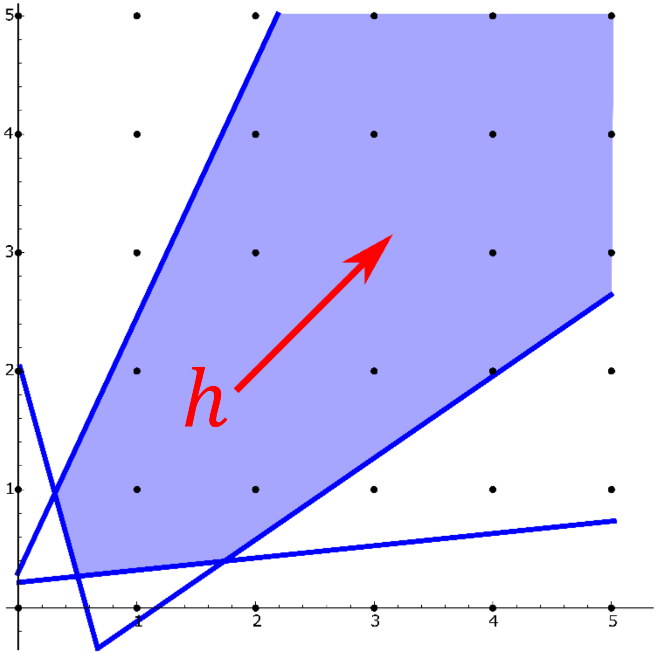

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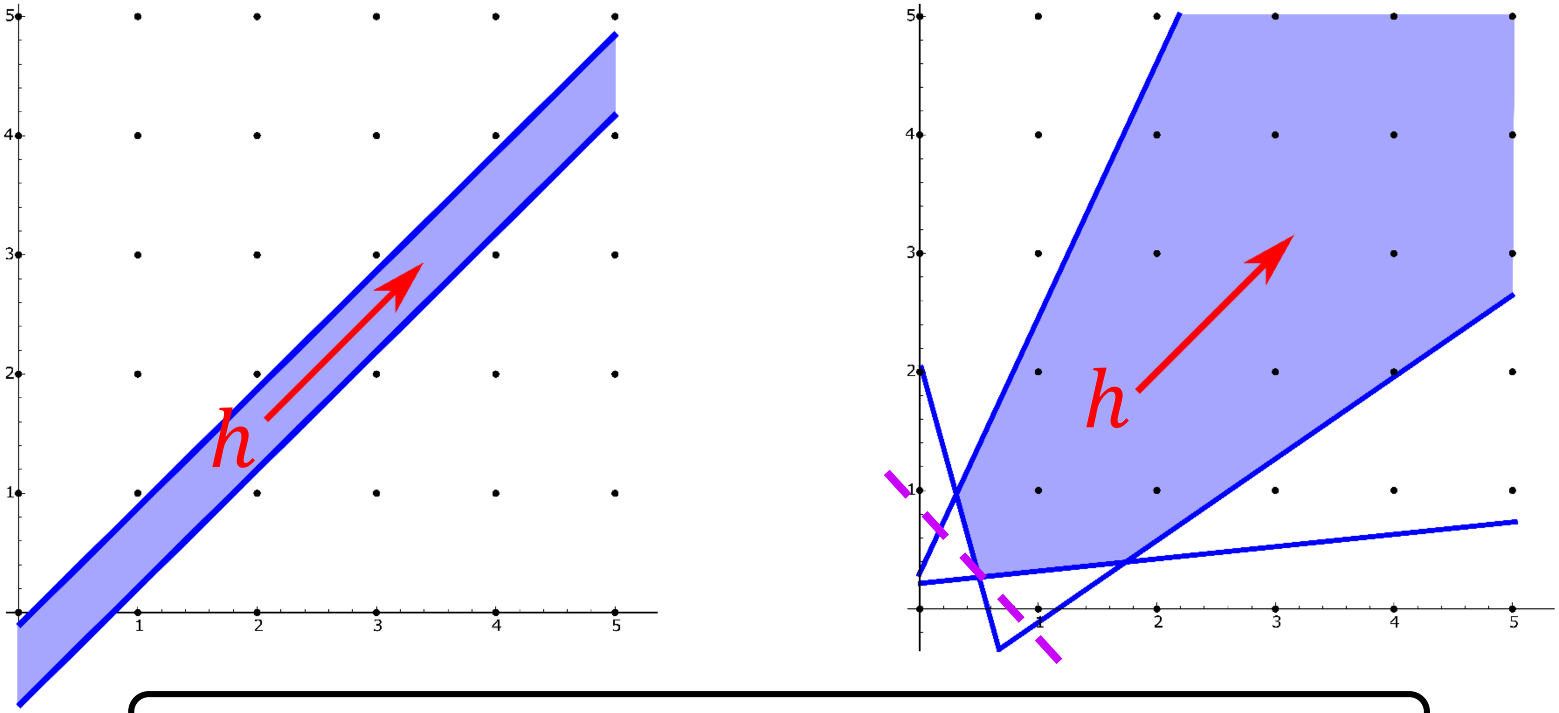

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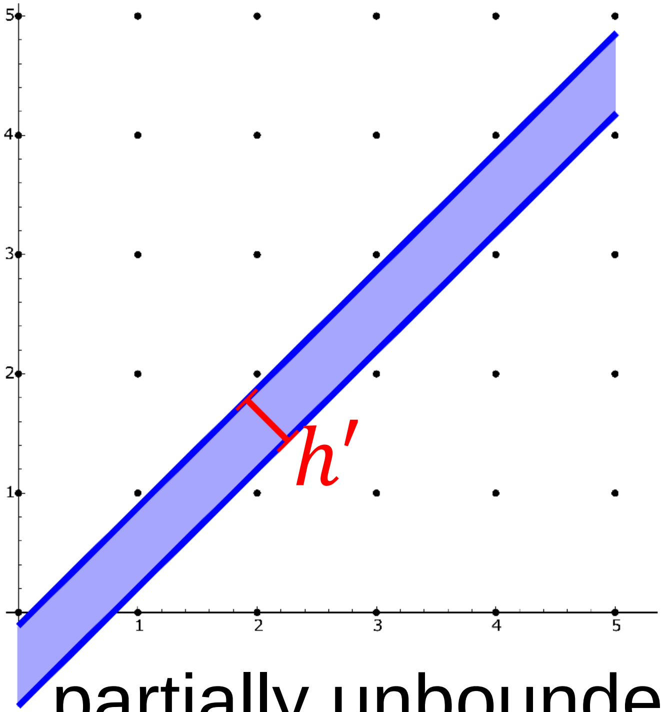
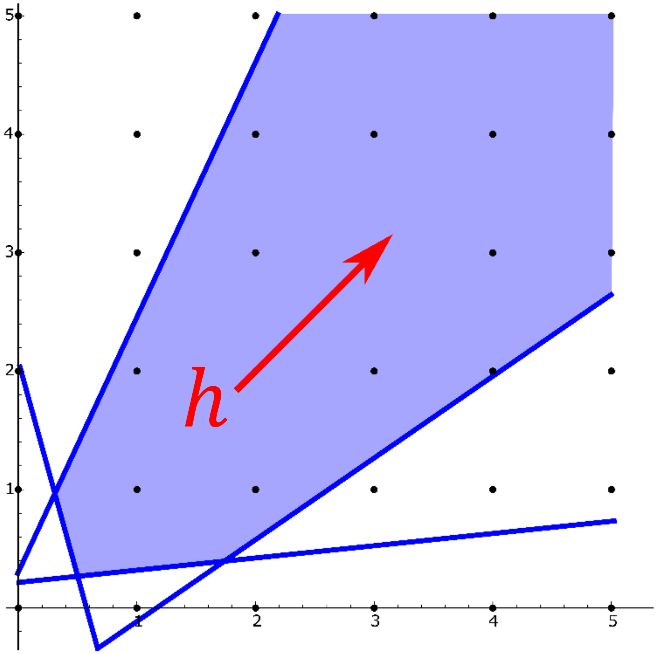
Unbounded Problems

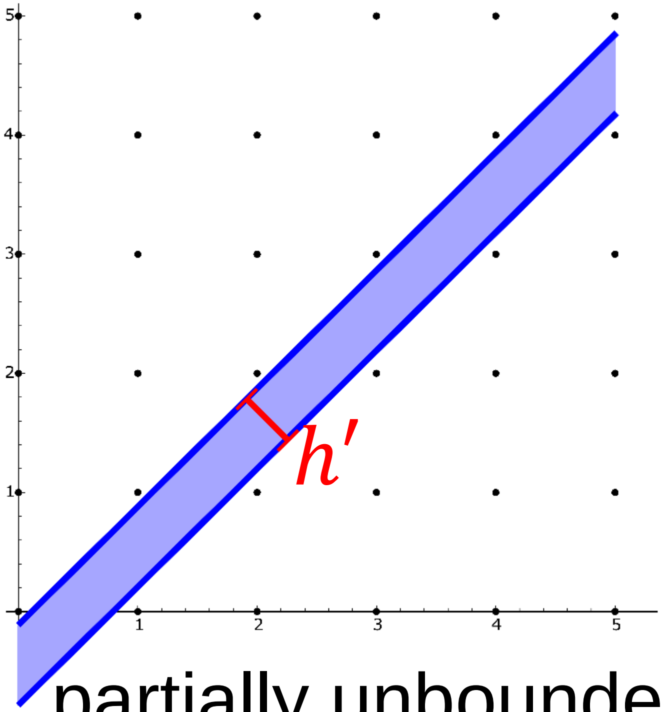


partially unbounded



partially unbounded:
both bounded and unbounded directions

Unbounded Problems



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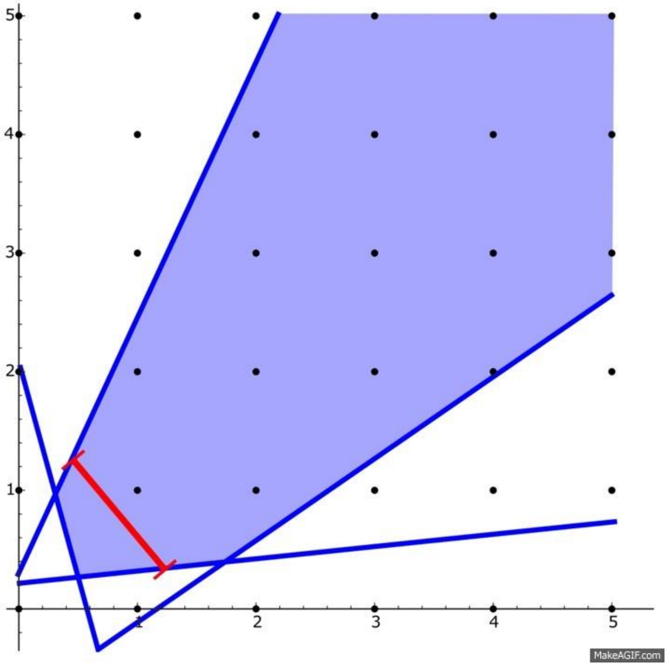
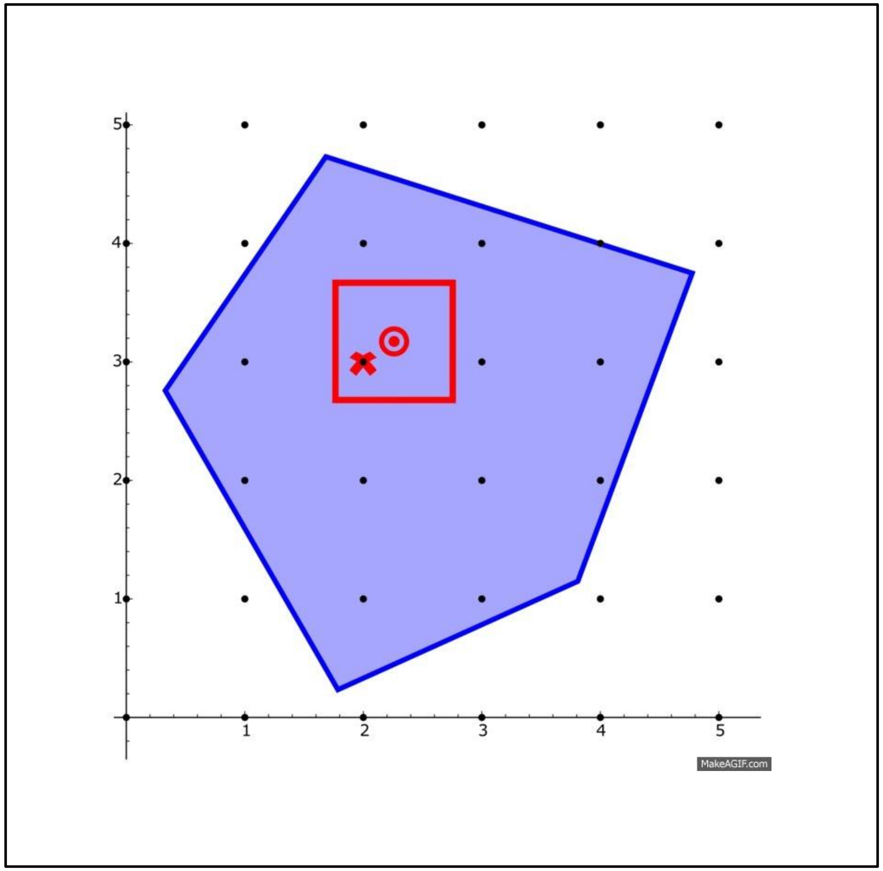


absolutely unbounded

absolutely unbounded:
only unbounded directions

Overview: Unit Cube Test

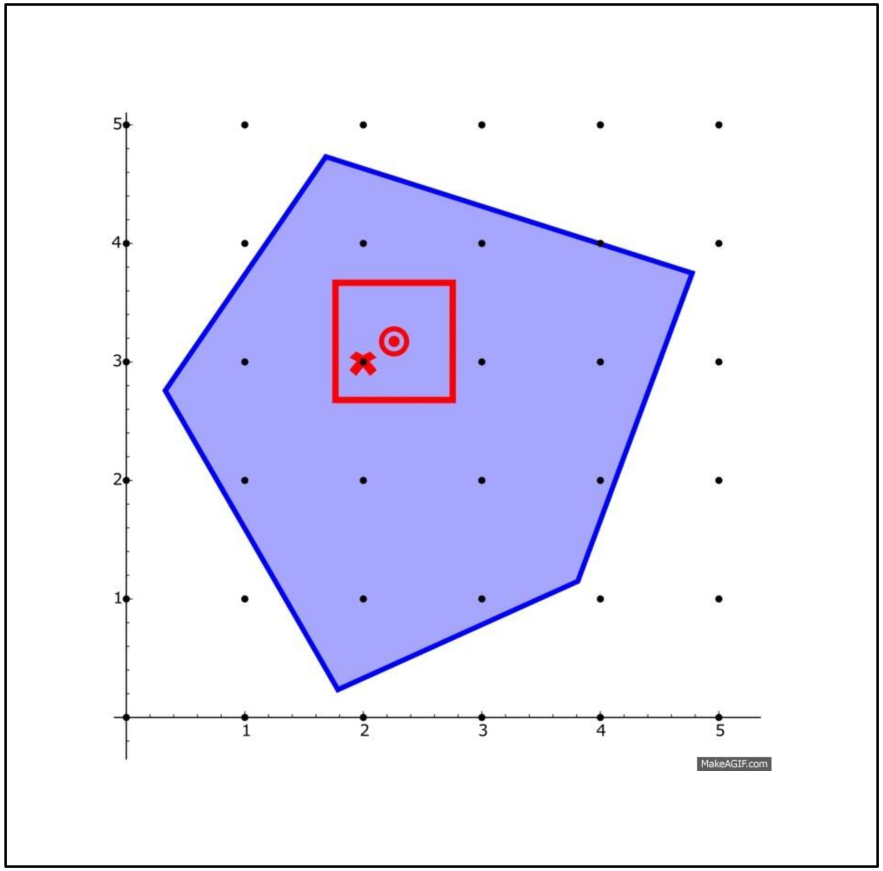
(IJCAR 2016)



for absolutely unbounded problems

Overview: Unit Cube Test

(IJCAR 2016)

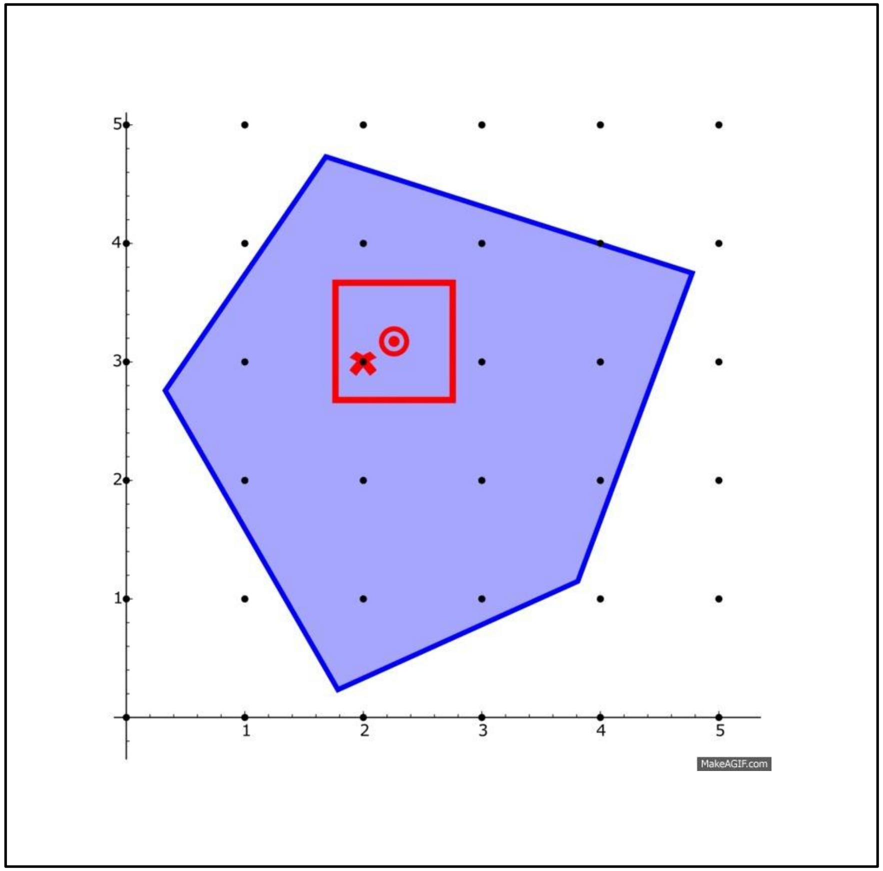


for absolutely unbounded problems

- **unit cube** guarantees integer solution

Overview: Unit Cube Test

(IJCAR 2016)



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- **unit cube** guarantees integer solution
- computable in **polynomial time**

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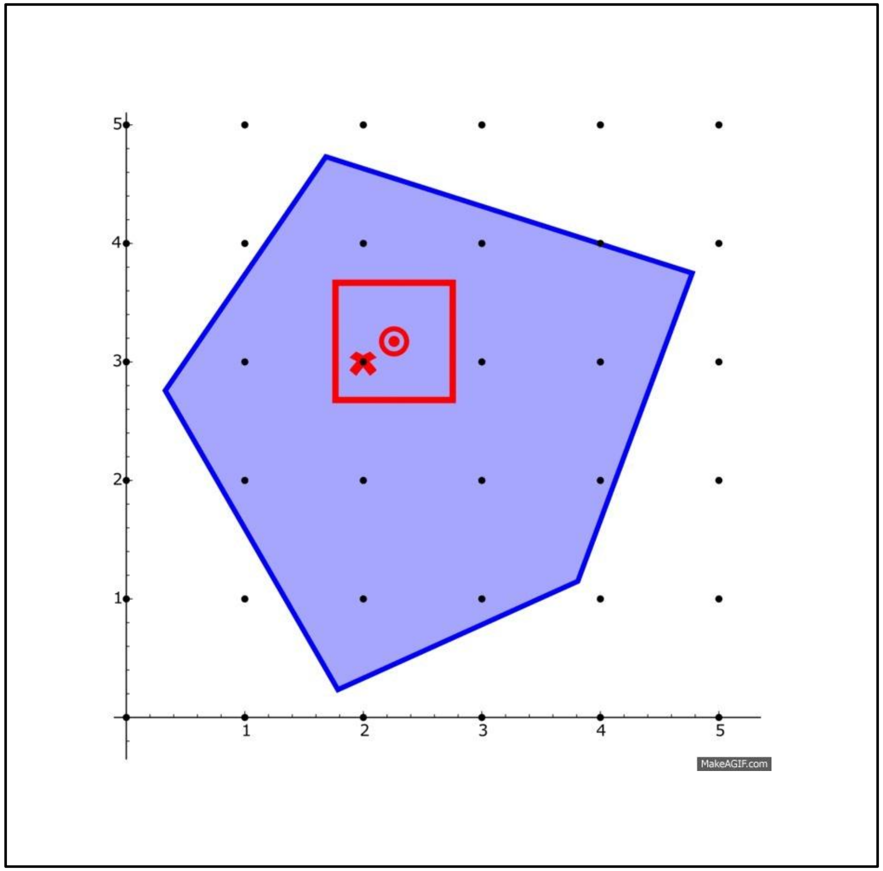
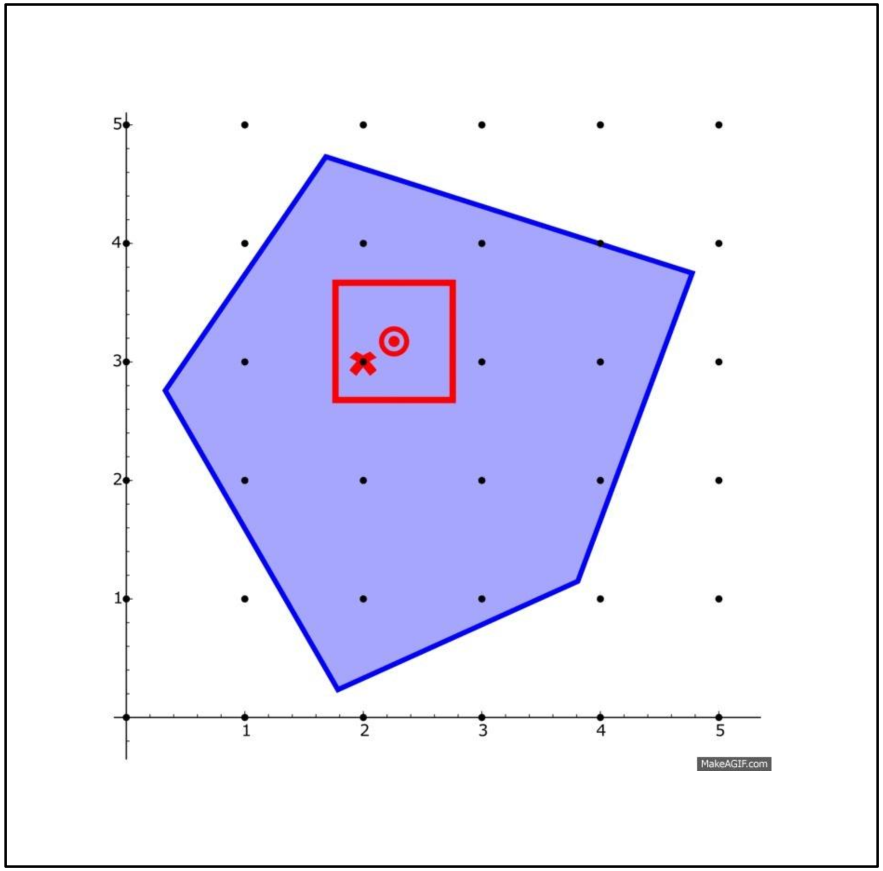


for absolutely unbounded problems

- **unit cube** guarantees integer solution
- computable in **polynomial time**
- **incremental**

Overview: Unit Cube Test

(IJCAR 2016)

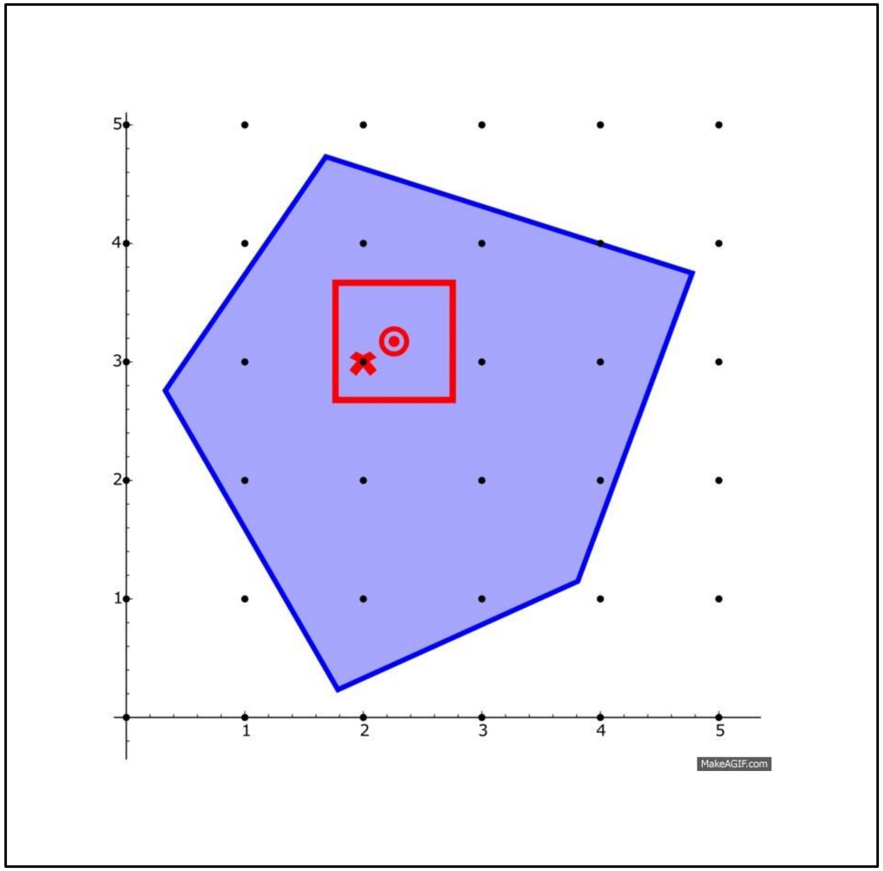


for absolutely unbounded problems

- **unit cube** guarantees integer solution
- computable in **polynomial time**
- **incremental**
- **not complete** in general

Overview: Unit Cube Test

(IJCAR 2016)

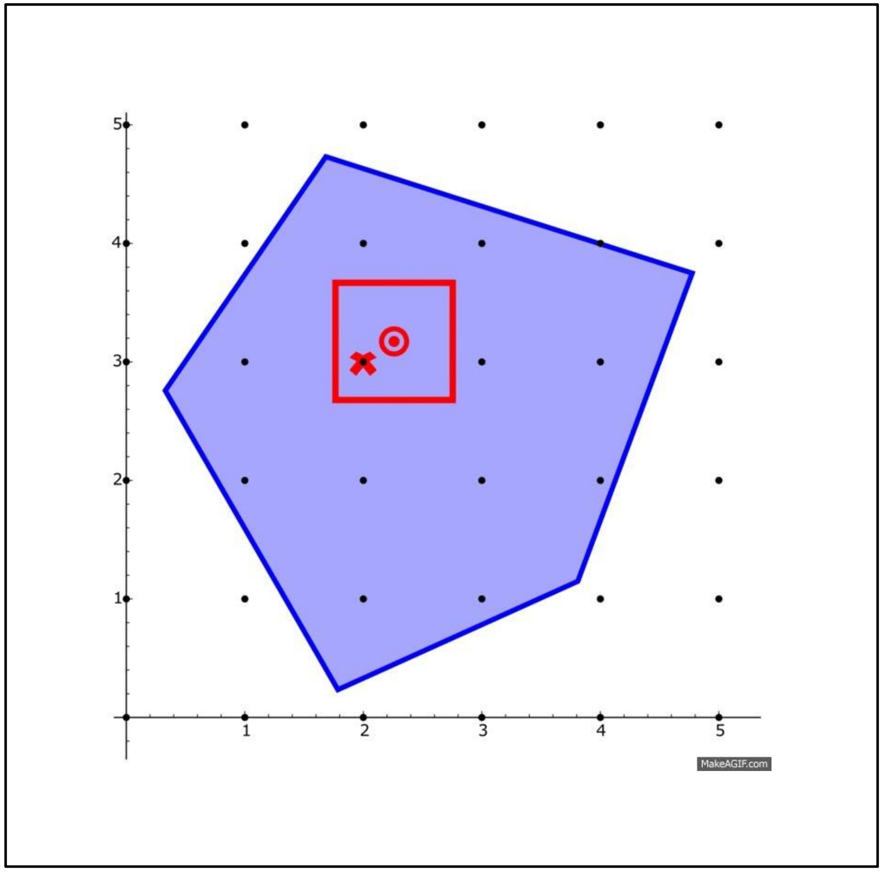


for absolutely unbounded problems

- **unit cube** guarantees integer solution
- computable in **polynomial time**
- **incremental**
- **not complete** in general
- always succeeds on abs. unbd. problems

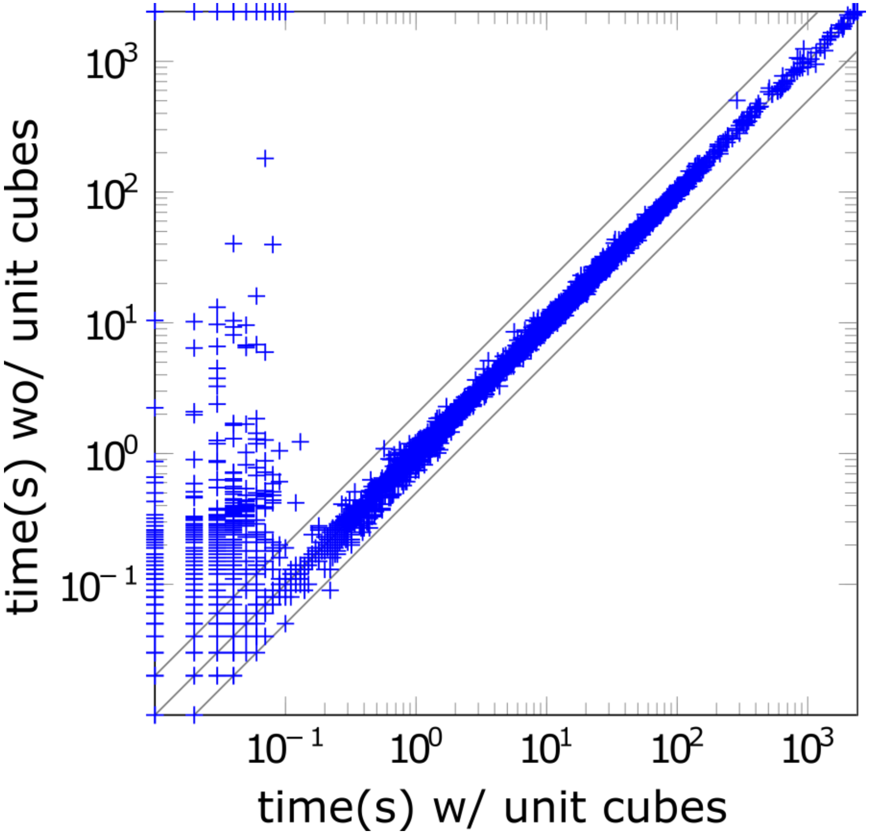
Results: Unit Cube Test

(IJCAR 2016)



for absolutely unbounded problems

QF_LIA (6947 problems)

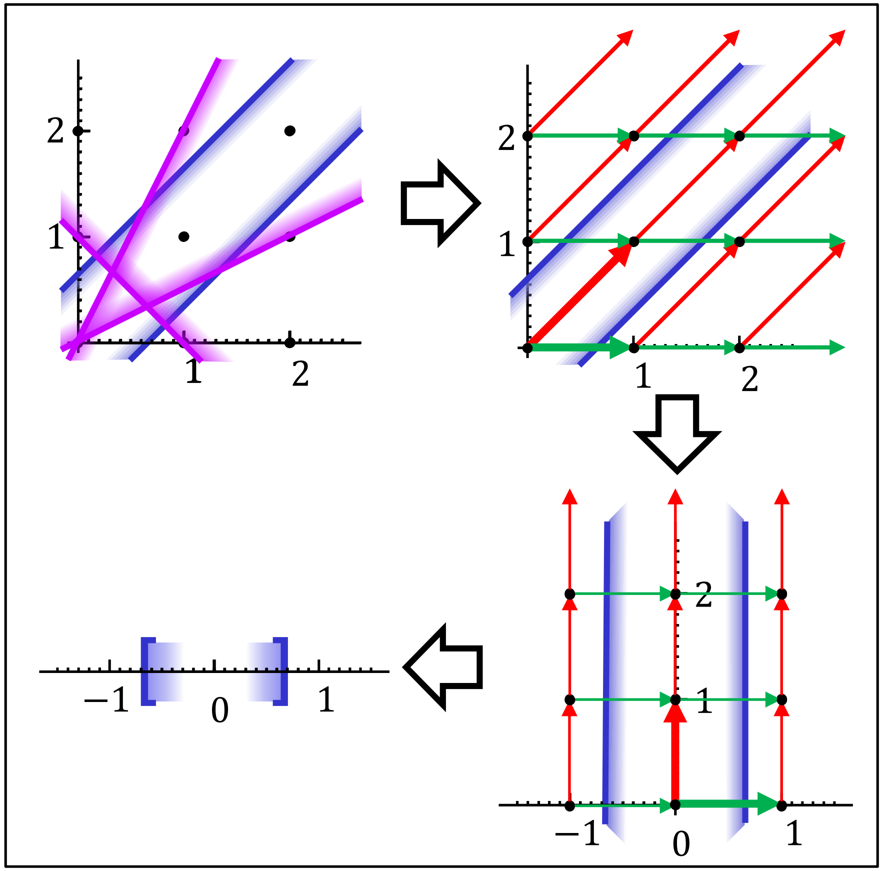


additional instances: 56

more than twice as fast: 705

Overview: Bounding Transformation

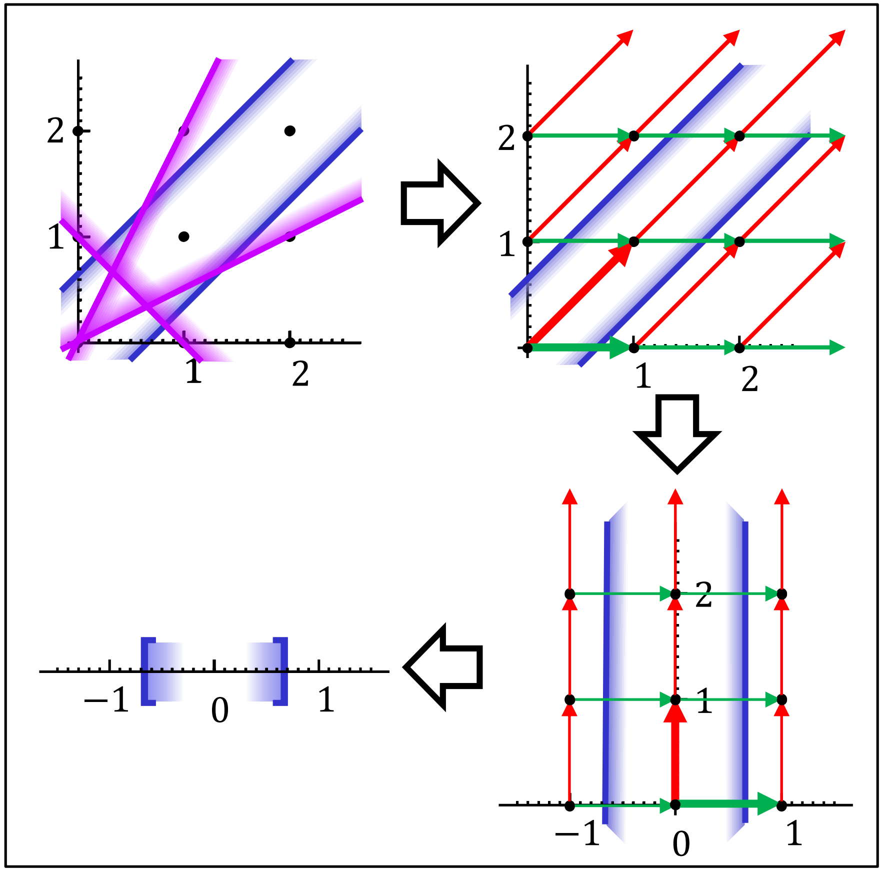
(IJCAR 2018)



for partially unbounded problems

Overview: Bounding Transformation

(IJCAR 2018)

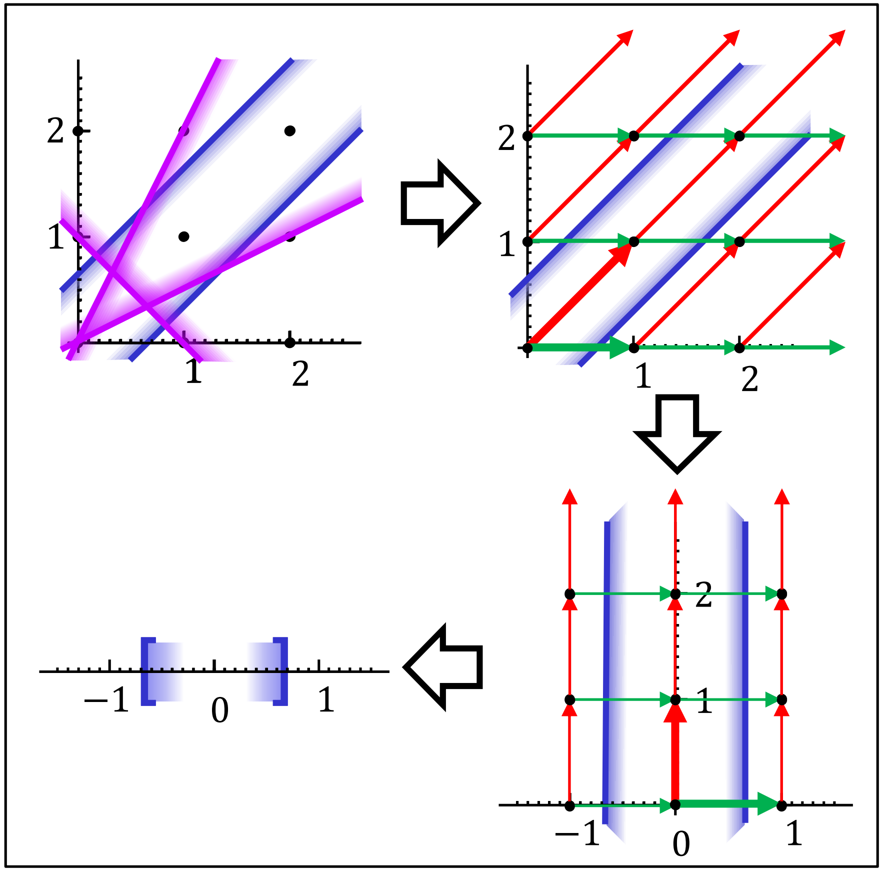


for partially unbounded problems

- transforms **unbounded into bounded** problems

Overview: Bounding Transformation

(IJCAR 2018)

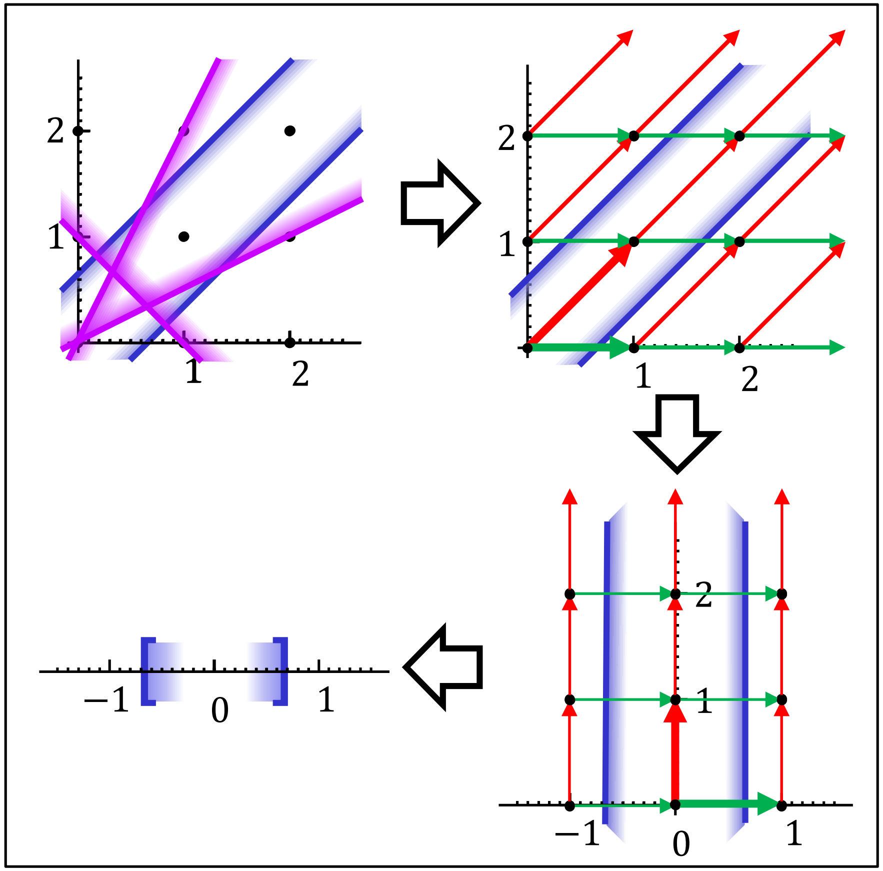


for partially unbounded problems

- transforms **unbounded into bounded** problems
- computable in **polynomial time**

Overview: Bounding Transformation

(IJCAR 2018)

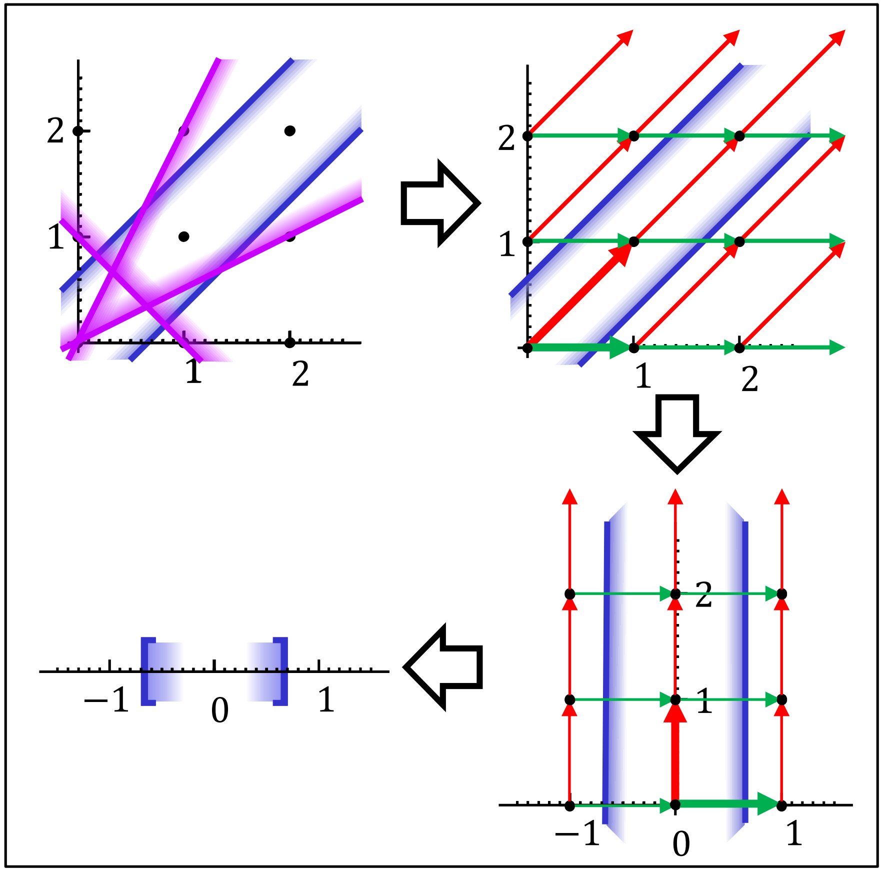


for partially unbounded problems

- transforms **unbounded into bounded** problems
- computable in **polynomial time**
- **solution & conflict conversion** (polynomial time)

Overview: Bounding Transformation

(IJCAR 2018)

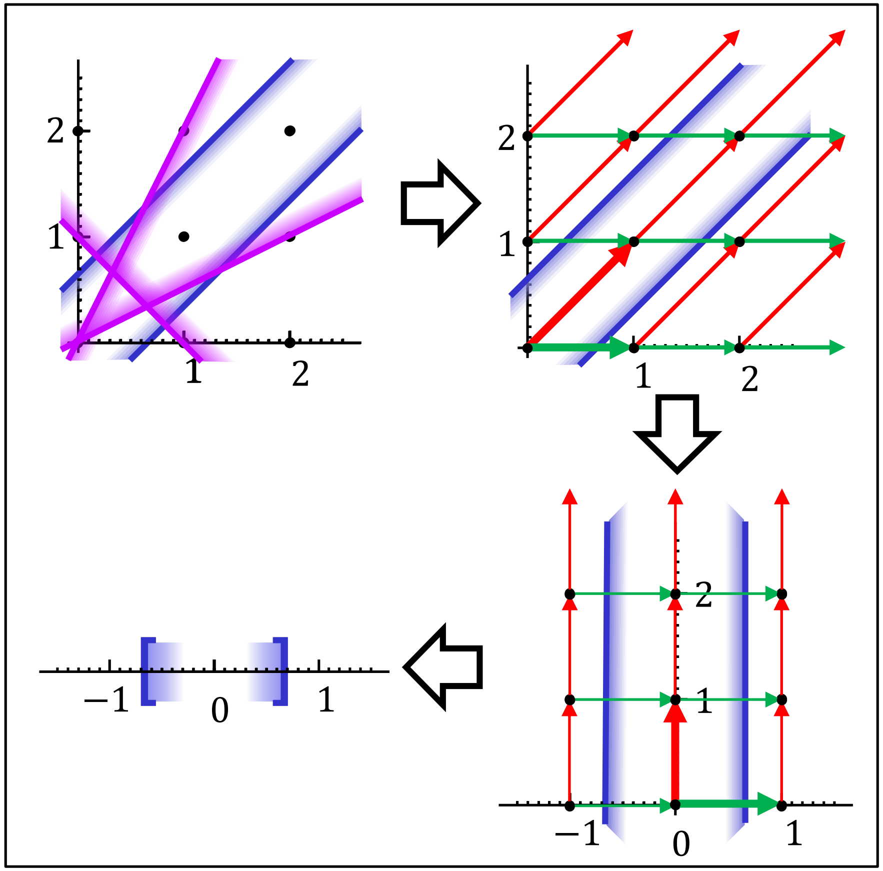


for partially unbounded problems

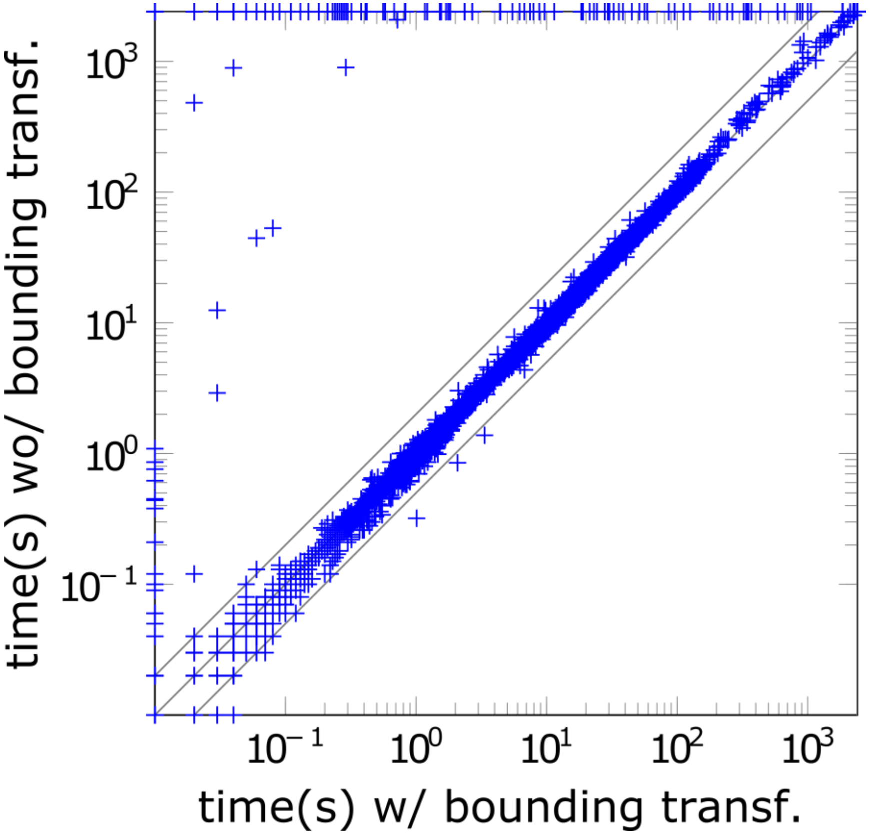
- transforms **unbounded into bounded** problems
- computable in **polynomial time**
- **solution & conflict conversion** (polynomial time)
- **incremental**

Results: Bounding Transformation

(IJCAR 2018)



QF_LIA (6947 problems)



for partially unbounded problems

additional instances: 169

more than twice as fast: 167

Preprocessing:

- if-then-else (reconstruction, lifting, simplification, bounding) [CVC4]
- pseudo-Boolean inequalities [CVC4]
- small CNF transformation [Weidenbach01]

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- if-then-else (reconstruction, lifting, simplification, bounding) [CVC4]
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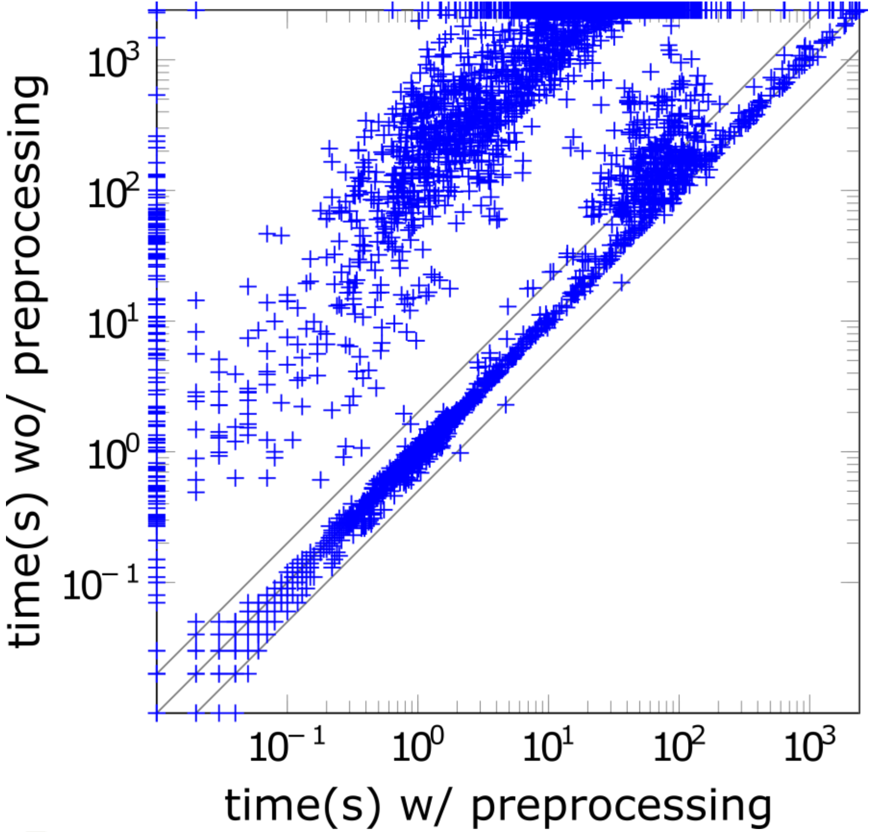
QF_LIA (6947 problems)



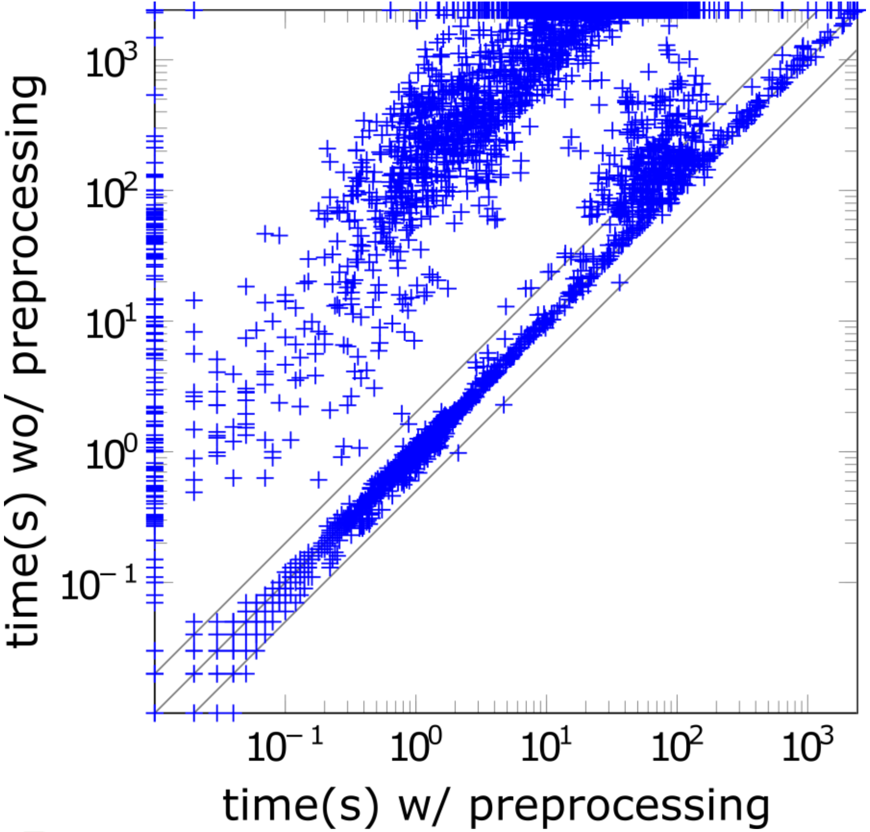
additional instances:1776

[…] invented by our team [...] invented & published by someone else […] never published but implemented

Preprocessing:

- if-then-else (reconstruction, lifting, simplification, bounding) [CVC4]
- pseudo-Boolean inequalities [CVC4]
- small CNF transformation [Weidenbach01]

QF_LIA (6947 problems)



additional instances:1776

Modular Arithmetic

Modular Arithmetic

$$2 \equiv_9 3 \cdot x \qquad\qquad for\ x \in \mathbb{Z}$$

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UNSAT

Modular Arithmetic

$$2 \equiv_9 3 \cdot x \qquad for\ x \in \mathbb{Z}$$

UNSAT

Proof by case distinction:

Modular Arithmetic

$$2 \equiv_9 3 \cdot x \qquad\qquad for\ x \in \mathbb{Z}$$

UNSAT

Proof by case distinction:

$x = 3 \cdot k$ for $k \in \mathbb{Z}$ $\quad 0 \equiv_9 3 \cdot (3 \cdot k)$

Modular Arithmetic

$$2 \equiv_9 3 \cdot x \qquad\qquad for\ x \in \mathbb{Z}$$

UNSAT

Proof by case distinction:

$x = 3 \cdot k$ for $k \in \mathbb{Z}$ $\quad 0 \equiv_9 3 \cdot (3 \cdot k)$

$x = 3 \cdot k + 1$ for $k \in \mathbb{Z}$ $\quad 3 \equiv_9 3 \cdot (3 \cdot k + 1)$

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Modular Arithmetic

$$2 \equiv_9 3 \cdot x \qquad\qquad \mathit{for}\ x \in \mathbb{Z}$$

UNSAT

Proof by case distinction:

$x = 3 \cdot k$ for $k \in \mathbb{Z}$ $\quad 0 \equiv_9 3 \cdot (3 \cdot k)$

$x = 3 \cdot k + 1$ for $k \in \mathbb{Z}$ $\quad 3 \equiv_9 3 \cdot (3 \cdot k + 1)$

$x = 3 \cdot k + 2$ for $k \in \mathbb{Z}$ $\quad 6 \equiv_9 3 \cdot (3 \cdot k + 2)$

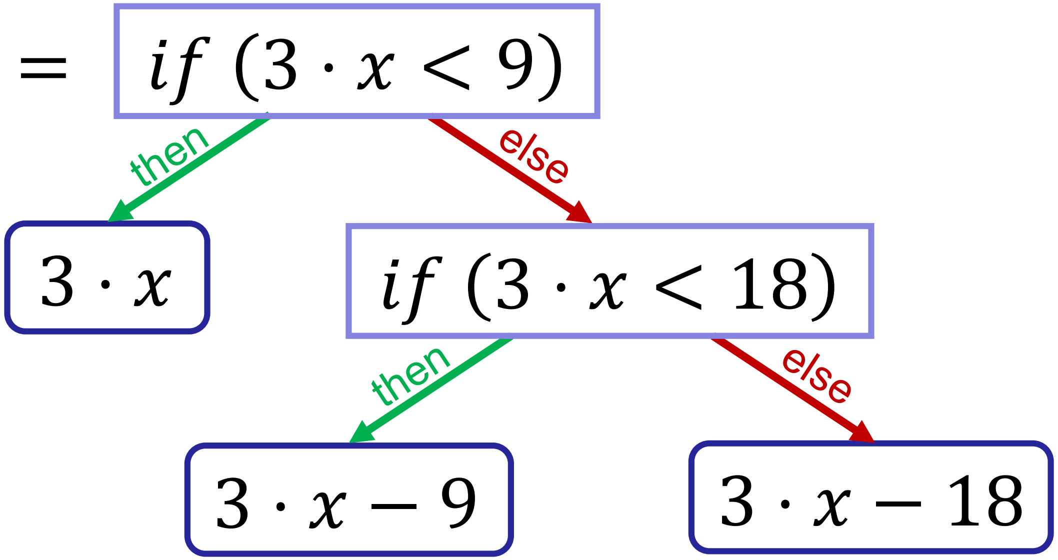
Modular Arithmetic via If-Then-Else

$$2 \equiv_9 3 \cdot x \qquad\qquad for\ x \in \mathbb{Z}$$

Modular Arithmetic via If-Then-Else

$$2 \equiv_9 3 \cdot x \qquad \mathit{for}\ x \in \mathbb{Z}$$

$$0 \leq x < 9 \quad \wedge \quad 2 =$$



$$\mathit{if}\ (3 \cdot x < 9)$$

$$3 \cdot x$$

$$\mathit{if}\ (3 \cdot x < 18)$$

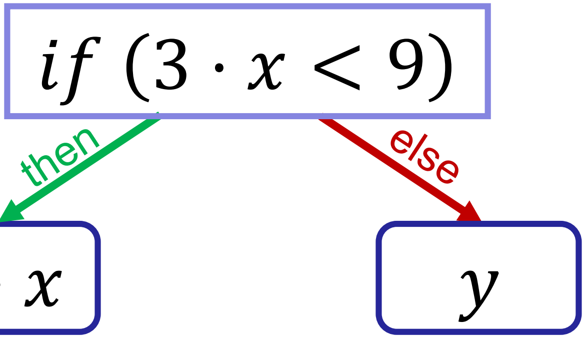
$$3 \cdot x - 9$$

$$3 \cdot x - 18$$

Modular Arithmetic via If-Then-Else

$$2 \equiv_9 3 \cdot x \qquad \mathit{for}\ x, y \in \mathbb{Z}$$

$$0 \leq x < 9 \quad \wedge \quad 2 =$$

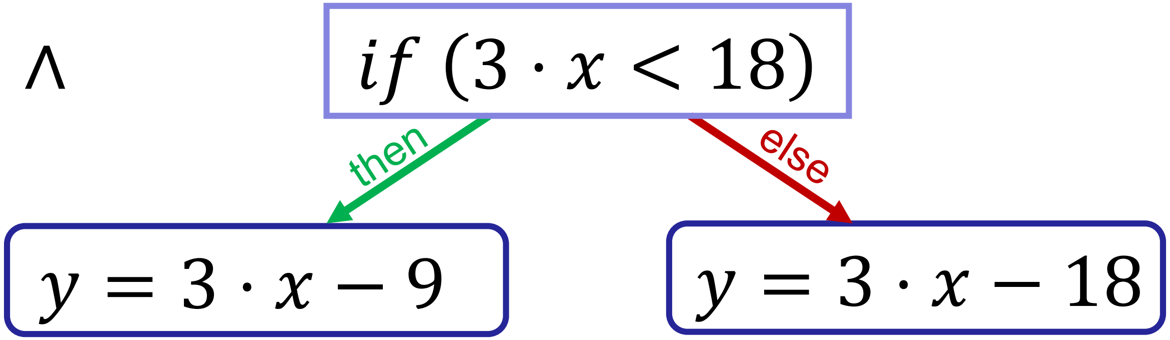


$if\ (3 \cdot x < 9)$

$3 \cdot x$

y

$$\wedge$$



$if\ (3 \cdot x < 18)$

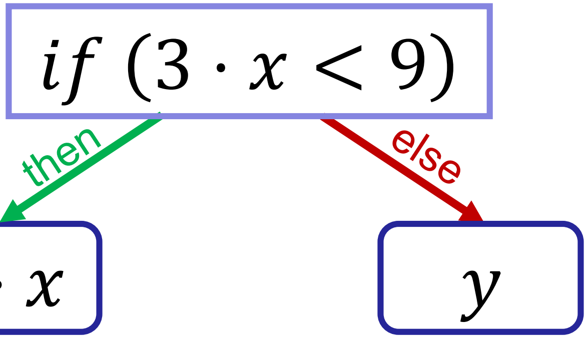
$y = 3 \cdot x - 9$

$y = 3 \cdot x - 18$

Modular Arithmetic via If-Then-Else

$$2 \equiv_9 3 \cdot x \qquad \mathit{for}\ x, y \in \mathbb{Z}$$

$$0 \le x < 9 \qquad \wedge \qquad 2 = \mathit{if}\ (3 \cdot x < 9)$$



$$\wedge \quad ((3 \cdot x < 18) \quad \vee \quad (y = 3 \cdot x - 18))$$

$$\wedge \quad (\neg(3 \cdot x < 18) \quad \vee \quad (y = 3 \cdot x - 9))$$

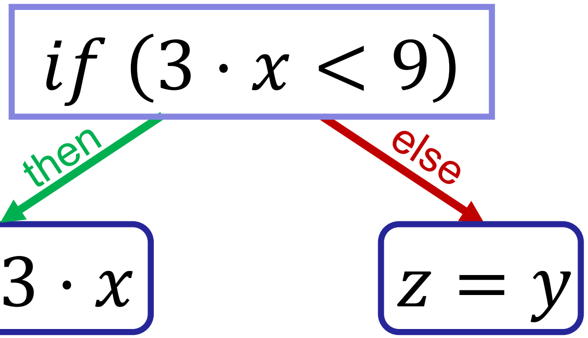
Modular Arithmetic via If-Then-Else

$$2 \equiv_9 3 \cdot x \qquad\qquad for\ x, y, z \in \mathbb{Z}$$

- -

$$0 \leq x < 9 \qquad \wedge \qquad 2 = z$$

$\wedge$



$if\ (3 \cdot x < 9)$

$z = 3 \cdot x$

$z = y$

$$\wedge \quad ((3 \cdot x < 18) \quad \vee \quad (y = 3 \cdot x - 18))$$

$$\wedge \quad (\neg(3 \cdot x < 18) \quad \vee \quad (y = 3 \cdot x - 9))$$

Modular Arithmetic via If-Then-Else

$$2 \equiv_9 3 \cdot x \qquad\qquad \mathit{for}\ x, y, z \in \mathbb{Z}$$

$$\wedge \quad 2 = z$$

$$\wedge \quad \boxed{(\boxed{(3 \cdot x < 9)} \ \vee \ \boxed{(z = 3 \cdot x)})}$$

$$\wedge \quad \boxed{(\boxed{\neg(3 \cdot x < 9)} \ \vee \ \boxed{(z = y)})}$$

$$\wedge \quad \boxed{(\boxed{(3 \cdot x < 18)} \ \vee \ \boxed{(y = 3 \cdot x - 18)})}$$

$$\wedge \quad \boxed{(\boxed{\neg(3 \cdot x < 18)} \ \vee \ \boxed{(y = 3 \cdot x - 9)})}$$

Modular Arithmetic via If-Then-Else

$$2 \equiv_9 3 \cdot x \qquad \qquad \mathit{for}\ x, y, z \in \mathbb{Z}$$

$$0 \le x < 9 \qquad \wedge \qquad 2 = z$$

- two new variables
- suboptimally connected

$$\wedge \quad \boxed{(\boxed{(3 \cdot x < 9)} \quad \vee \quad \boxed{(z = 3 \cdot x)})}$$

$$\wedge \quad \boxed{(\boxed{\neg(3 \cdot x < 9)} \quad \vee \quad \boxed{(z = y)})}$$

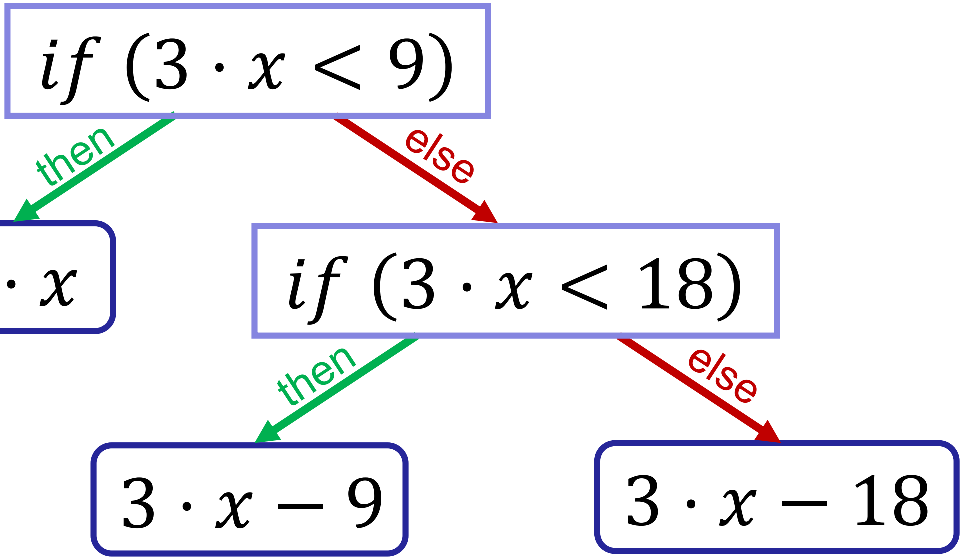
$$\wedge \quad \boxed{(\boxed{(3 \cdot x < 18)} \quad \vee \quad \boxed{(y = 3 \cdot x - 18)})}$$

$$\wedge \quad \boxed{(\boxed{\neg(3 \cdot x < 18)} \quad \vee \quad \boxed{(y = 3 \cdot x - 9)})}$$

If-Then-Else: Shared Monomial Lifting

$$2 \equiv_9 3 \cdot x \qquad for\ x \in \mathbb{Z}$$

$$\wedge \quad 2 =$$



$if\ (3 \cdot x < 9)$

then: $3 \cdot x$

else: $if\ (3 \cdot x < 18)$

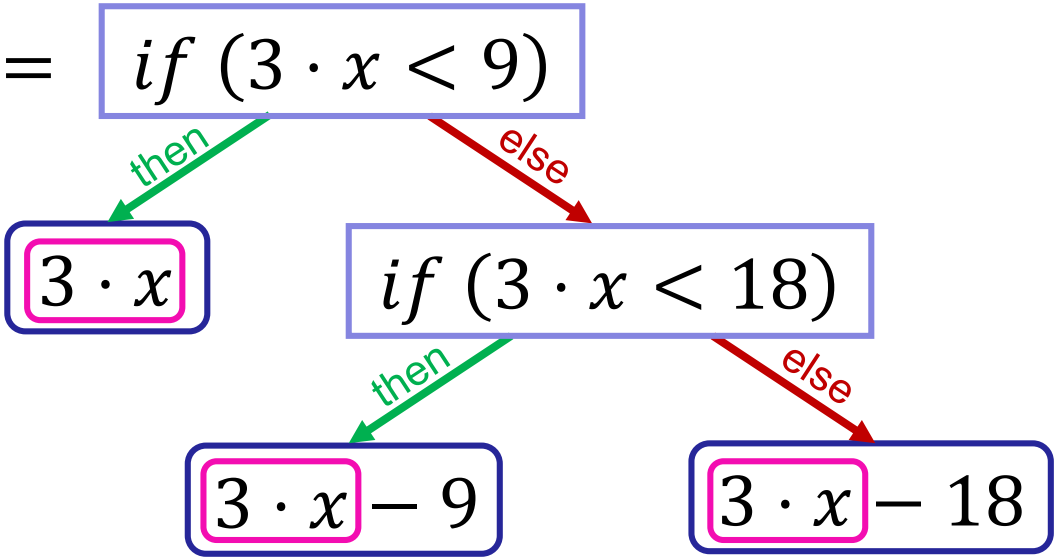
then: $3 \cdot x - 9$

else: $3 \cdot x - 18$

If-Then-Else: Shared Monomial Lifting

$$2 \equiv_9 3 \cdot x \qquad for\ x \in \mathbb{Z}$$

$0 \leq x < 9 \quad \wedge \quad 2 =$



$if\ (3 \cdot x < 9)$

$\boxed{3 \cdot x}$

$if\ (3 \cdot x < 18)$

$\boxed{3 \cdot x} - 9$

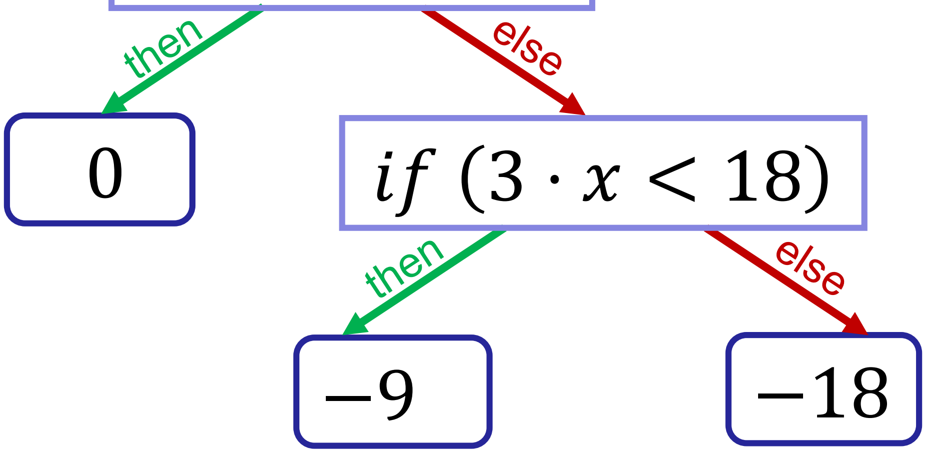
$\boxed{3 \cdot x} - 18$

All share the monomial $\boxed{3 \cdot x}$!

If-Then-Else: Shared Monomial Lifting

$$2 \equiv_9 3 \cdot x \qquad\qquad for\ x \in \mathbb{Z}$$

$$\wedge \quad 2 = \boxed{3 \cdot x} + \boxed{if\ (3 \cdot x < 9)}$$



then

else

0

$if\ (3 \cdot x < 18)$

then

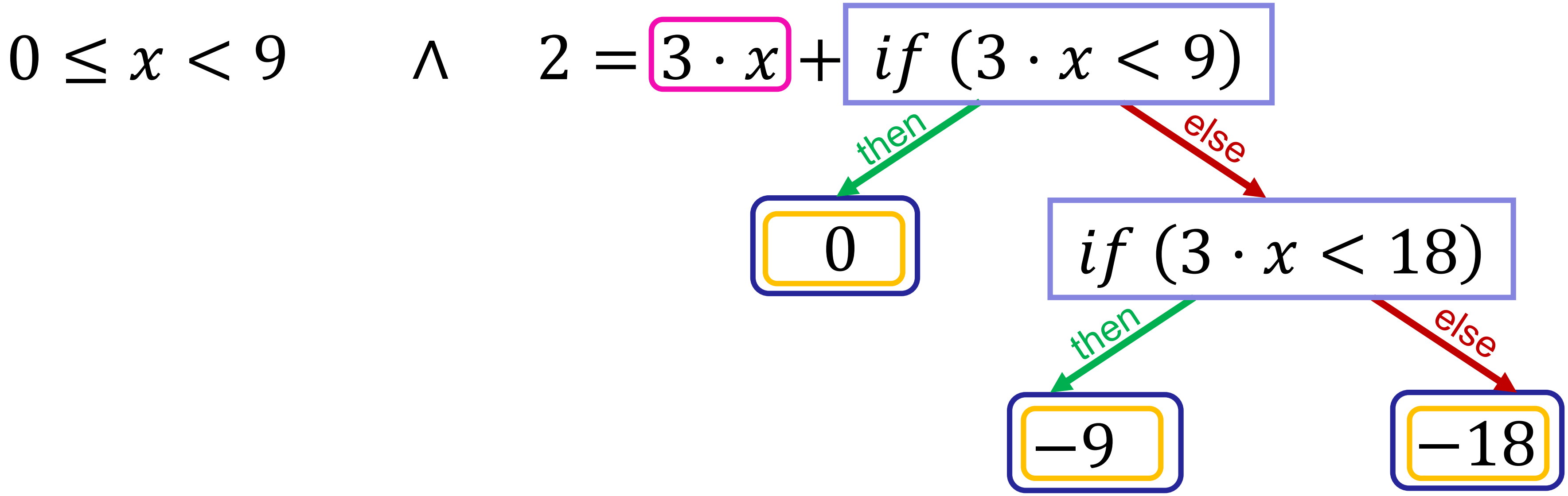
else

-9

-18

If-Then-Else: Shared Monomial Lifting

$$2 \equiv_9 3 \cdot x \qquad \qquad for\ x \in \mathbb{Z}$$



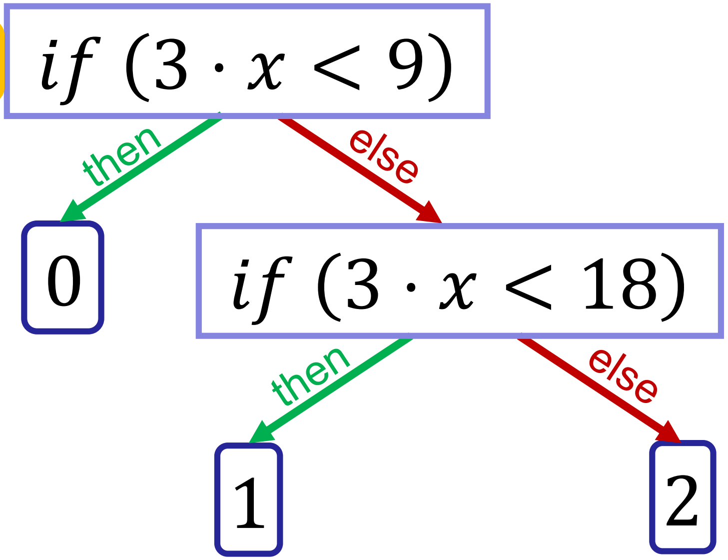
$$0 \le x < 9 \quad \wedge \quad 2 = \boxed{3 \cdot x} + \boxed{if\ (3 \cdot x < 9)}$$

All divisible by $\boxed{-9}$!

If-Then-Else: Shared Monomial Lifting

$$2 \equiv_9 3 \cdot x \qquad for\ x \in \mathbb{Z}$$

$$0 \le x < 9 \qquad \wedge \qquad 2 = \boxed{3 \cdot x}\boxed{- 9 \cdot}\boxed{if\ (3 \cdot x < 9)}$$

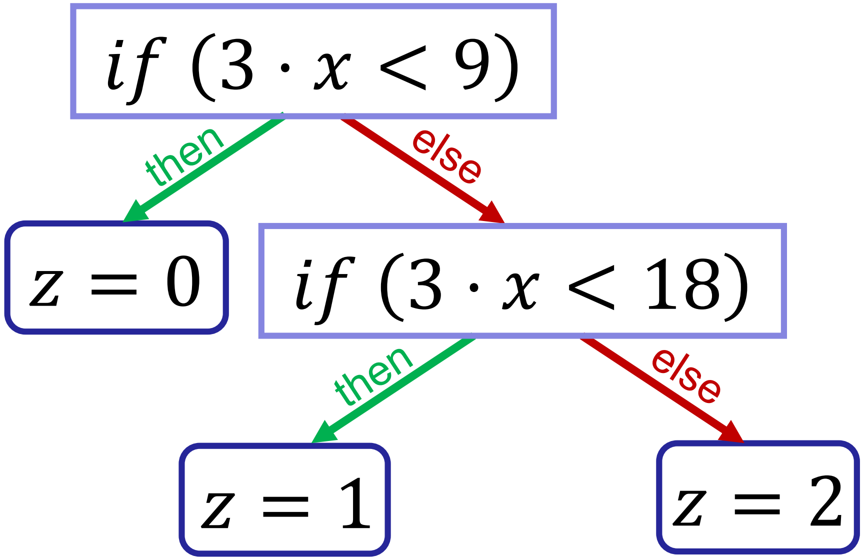


If-Then-Else: Bounding

$$2 \equiv_9 3 \cdot x \qquad\qquad for\ x, z \in \mathbb{Z}$$

$$\wedge \quad 2 = \boxed{3 \cdot x} \boxed{- 9 \cdot} z$$

$\wedge$



$if\ (3 \cdot x < 9)$

$z = 0$

$if\ (3 \cdot x < 18)$

$z = 1$

$z = 2$

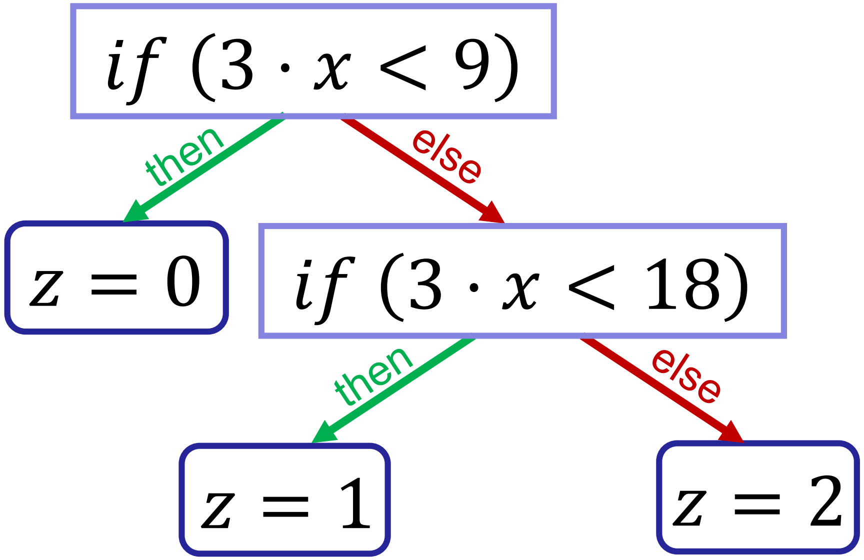
If-Then-Else: Bounding

$$2 \equiv_9 3 \cdot x \qquad \mathit{for}\ x, z \in \mathbb{Z}$$

- -

$$0 \leq x < 9 \quad \wedge \quad 2 = \boxed{3 \cdot x}\boxed{- 9 \cdot} z \quad \wedge \quad 0 \leq z \leq 2$$

$\wedge$



$\mathit{if}\ (3 \cdot x < 9)$

then

else

$z = 0$

$\mathit{if}\ (3 \cdot x < 18)$

then

else

$z = 1$

$z = 2$

If-Then-Else: Preprocessing

$$2 \equiv_9 3 \cdot x \qquad \qquad for\ x, z \in \mathbb{Z}$$

$$0 \le x < 9 \quad \wedge \quad 2 = \boxed{3 \cdot x} \boxed{- 9 \cdot} z \quad \wedge \quad 0 \le z \le 2$$

$$\wedge \ \left(\neg(3 \cdot x < 9) \ \vee \ \boxed{z = 0}\right)$$

$$\wedge \ \left((3 \cdot x < 9) \ \vee \neg(3 \cdot x < 18) \ \vee \ \boxed{z = 1}\right)$$

$$\wedge \ \left(\neg(3 \cdot x < 18) \ \vee \ \boxed{z = 2}\right)$$

If-Then-Else: Preprocessing

$$2 \equiv_9 3 \cdot x \qquad for\ x, z \in \mathbb{Z}$$

$$0 \leq x < 9 \quad \wedge \quad \boxed{2 = 3 \cdot x - 9 \cdot z} \quad \wedge \quad 0 \leq z \leq 2$$

$$\wedge\ (\neg(3 \cdot x < 9)\ \vee\ z = 0\ )$$

$$\wedge\ \big((3 \cdot x < 9)\ \vee \neg(3 \cdot x < 18)\ \vee\ z = 1\big)$$

$$\wedge\ (\neg(3 \cdot x < 18)\ \vee\ z = 2\ )$$

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If-Then-Else: Preprocessing

$$2 \equiv_9 3 \cdot x \qquad \mathit{for}\ x, z \in \mathbb{Z}$$

$$0 \leq x < 9 \quad \wedge \quad 2 \leq 3 \cdot x - 9 \cdot z \quad \wedge \quad 0 \leq z \leq 2$$

$$\wedge \quad 2 \geq 3 \cdot x - 9 \cdot z$$

$$\wedge \ (\neg(3 \cdot x < 9) \ \vee \ z = 0\ )$$

$$\wedge \ \big((3 \cdot x < 9) \ \vee \neg(3 \cdot x < 18) \ \vee \ z = 1\big)$$

$$\wedge \ (\neg(3 \cdot x < 18) \ \vee \ z = 2\ )$$

If-Then-Else: Preprocessing

$$\begin{array}{c} 2 \equiv_9 3 \cdot x \qquad\qquad \mathit{for}\ x, z \in \mathbb{Z} \\ \hdashline \end{array}$$

$$0 \leq x < 9 \quad \boxed{\begin{array}{l} \wedge \quad \frac{2}{3} \leq 1 \cdot x - 3 \cdot z \\[2em] \wedge \quad \frac{2}{3} \geq 1 \cdot x - 3 \cdot z \end{array}} \quad \wedge \quad 0 \leq z \leq 2$$

$$\wedge \ (\neg(3 \cdot x < 9) \ \vee \ z = 0\ )$$

$$\wedge \ \big((3 \cdot x < 9) \vee \neg(3 \cdot x < 18) \ \vee \ z = 1\big)$$

$$\wedge \ (\neg(3 \cdot x < 18) \ \vee \ z = 2\ )$$

If-Then-Else: Preprocessing

$$2 \equiv_9 3 \cdot x \qquad for\ x, z \in \mathbb{Z}$$

$$0 \le x < 9 \quad \wedge \left\lceil \frac{2}{3} \right\rceil \le 1 \cdot x - 3 \cdot z \quad \wedge \quad 0 \le z \le 2$$

$$\wedge \left\lfloor \frac{2}{3} \right\rfloor \ge 1 \cdot x - 3 \cdot z$$

$$\wedge\ (\neg(3 \cdot x < 9)\ \vee\ z = 0\ )$$

$$\wedge\ \big((3 \cdot x < 9)\ \vee \neg(3 \cdot x < 18)\ \vee\ z = 1\big)$$

$$\wedge\ (\neg(3 \cdot x < 18)\ \vee\ z = 2\ )$$

If-Then-Else: Preprocessing

$$2 \equiv_9 3 \cdot x \qquad \qquad for\ x, z \in \mathbb{Z}$$

$$\wedge \quad 1 \leq 1 \cdot x - 3 \cdot z \qquad \wedge \quad 0 \leq z \leq 2$$

$$\wedge \quad 0 \geq 1 \cdot x - 3 \cdot z$$

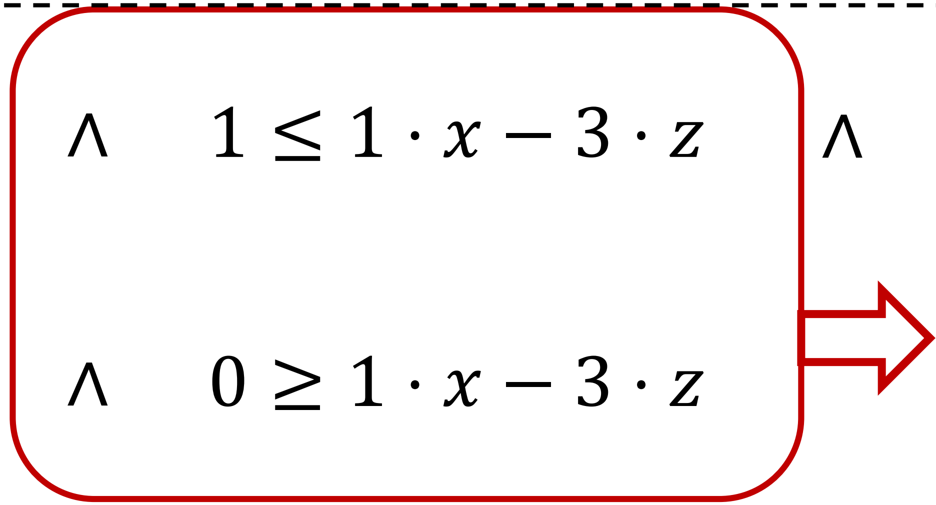
$$\wedge\ (\neg(3 \cdot x < 9)\ \vee\ z = 0\ )$$

$$\wedge\ \big((3 \cdot x < 9)\ \vee \neg(3 \cdot x < 18)\ \vee\ z = 1\big)$$

$$\wedge\ (\neg(3 \cdot x < 18)\ \vee\ z = 2\ )$$

If-Then-Else: Preprocessing

$$2 \equiv_9 3 \cdot x \qquad \mathit{for}\ x, z \in \mathbb{Z}$$



$$0 \leq x < 9 \quad \wedge \quad 1 \leq 1 \cdot x - 3 \cdot z \quad \wedge \quad 0 \leq z \leq 2$$

$$1 \leq 0$$

$$\wedge \quad 0 \geq 1 \cdot x - 3 \cdot z$$

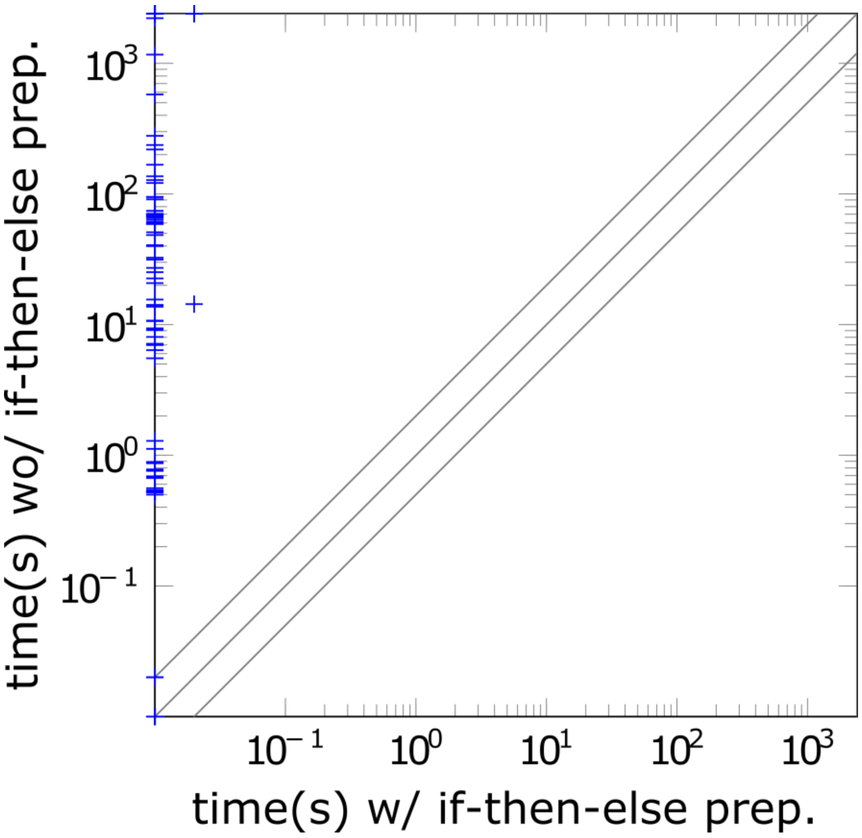
$$\wedge\ (\neg(3 \cdot x < 9)\ \vee\ z = 0\ )$$

$$\wedge\ \big((3 \cdot x < 9)\ \vee \neg(3 \cdot x < 18)\ \vee\ z = 1\big)$$

$$\wedge\ (\neg(3 \cdot x < 18)\ \vee\ z = 2\ )$$

If-Then-Else: Preprocessing

rings (294 problems)

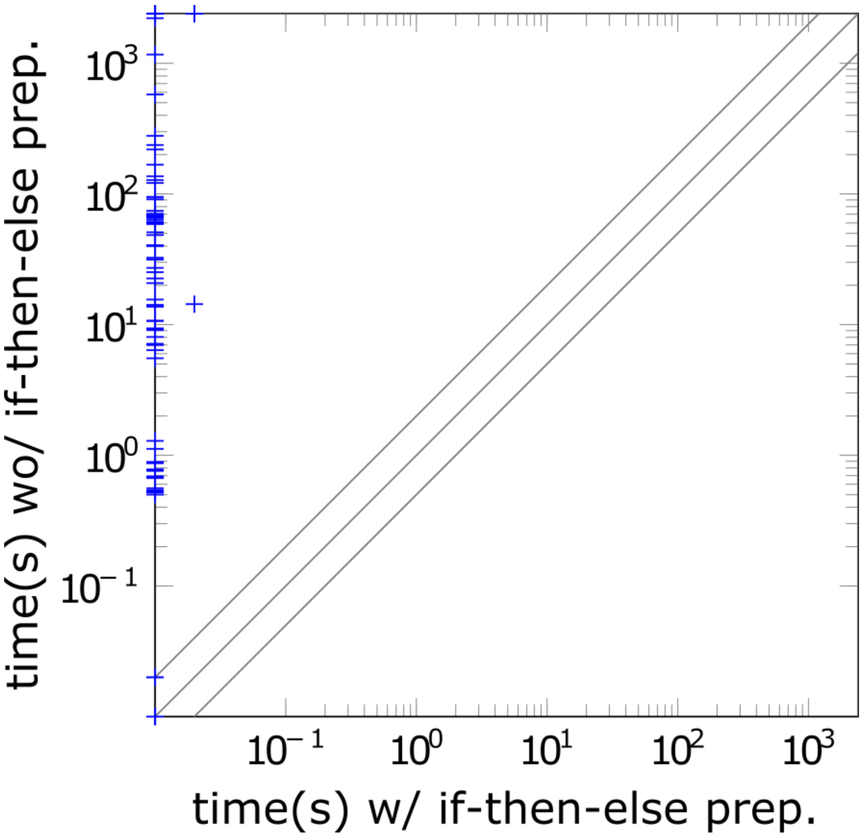


additional instances:157

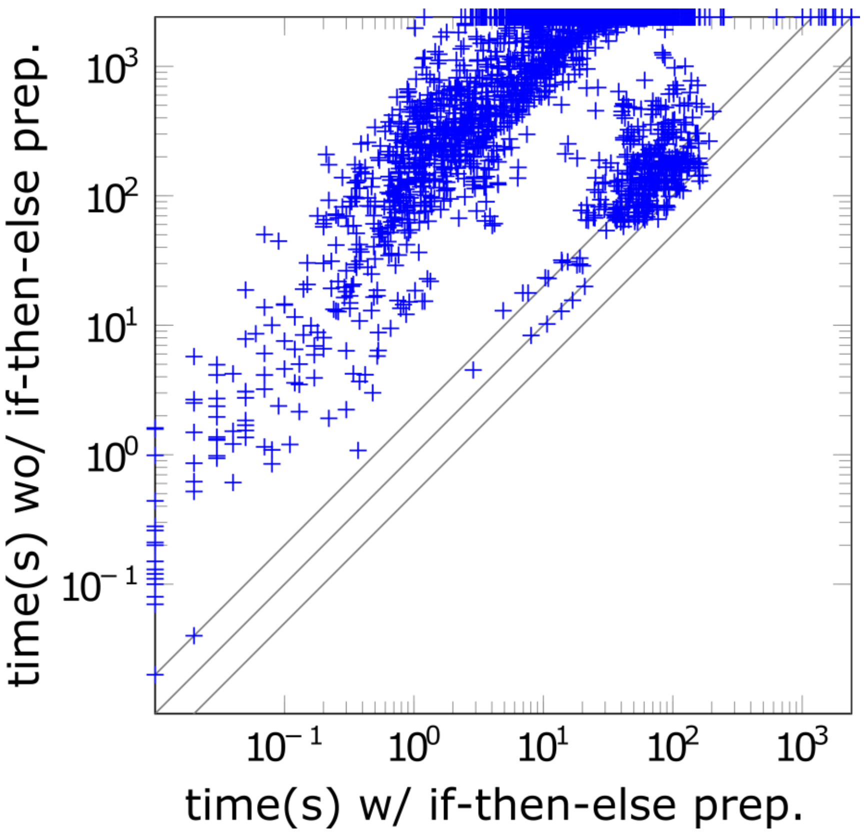
Techniques: shared monomial lifting,
ite bounding, (ite reconstruction)

If-Then-Else: Preprocessing

rings (294 problems)



nec_smt (2800 problems)



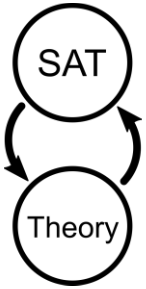
additional instances:157

additional instances: 1422

Techniques: shared monomial lifting,
ite bounding, (ite reconstruction)

Techniques: constant-ite simplification,
conjunctive-ite compression

[…] invented by our team […] invented & published by someone else […] never published but implemented



SAT and theory interaction:

- weakened early pruning [Sebastiani07]
- unate propagations and bound refinements [Dutertre06]
- decision recommendations [Yices]

Theory solver extensions:

- unit cube test [Bromberger16]
- bounding transformation [Bromberger18]
- simple rounding and bound propagation [Schrijver86]

Data-structure improvements:

- priority queue for pivot selection [pretty much everyone]
- integer coefficients instead of rational coefficients [veriT]
- backup instead of recalculation [pretty much everyone]

Preprocessing:

- if-then-else (reconstruction, lifting, simplification, bounding) [CVC4]
- pseudo-Boolean inequalities [CVC4]
- small CNF transformation [Weidenbach01]

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