Formalizing Bachmair and Ganzinger's Ordered Resolution Prover

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Formalizing Automated Reasoning in Proof Assistants

Theorem

Proof assistants are mature enough to be used by researchers in AR.

Proof

We have formalized

- Bachmair and Ganzinger's resolution prover from Handbook of AR
- its soundness theorem
- its completeness theorem.

Corollary

We contribute to a growing library of formalized results in AR.

(Which makes the theorem even more true.)

IsaFoL: Isabelle Formalization of Logic

Framework and methodology

for reasoning about AR in Isabelle.

Adoption by AR researchers

as a convenient way to develop metatheory.

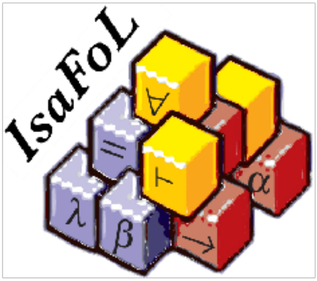
ITP benefits from ATP

… **why not the other way round?**

Isabelle @ RTA, Coq @ POPL

… **now Isabelle @ IJCAR.**

Eat our own dog food!



bitbucket.org/isafol

Ordered Ground Resolution

Ordered resolution with selection

$$\frac{C_1 \vee A_1 \vee \cdots \vee A_1 \quad \ldots \quad C_n \vee A_n \vee \cdots \vee A_n \quad \neg A_1 \vee \cdots \vee \neg A_n \vee D}{C_1 \vee \cdots \vee C_n \vee D}$$

where

(i) either the subclause $\neg A_1 \vee \cdots \vee \neg A_n$ is selected by S in D, or else $S(D)$ is empty, $n = 1$, and A_1 is maximal with respect to D,

(ii) each atom A_i is strictly maximal with respect to C_i, and

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Ordered Ground Resolution

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```
inductive
 ord_resolve :: "'a clause list ⇒ 'a clause ⇒ 'a clause ⇒ bool"
 where
  "length (CAs :: 'a clause list) = n ⟹
   length (Cs  :: 'a clause list) = n ⟹
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   ∀i < n. (CAs ! i) = (Cs ! i + (poss (AAs ! i))) ⟹
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} Length of lists

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Length of lists

Composition of clauses

Side conditions

Side Conditions

Ordered resolution with selection

$$\frac{C_1 \vee A_1 \vee \cdots \vee A_1 \quad \ldots \quad C_n \vee A_n \vee \cdots \vee A_n \quad \neg A_1 \vee \cdots \vee \neg A_n \vee D}{C_1 \vee \cdots \vee C_n \vee D}$$

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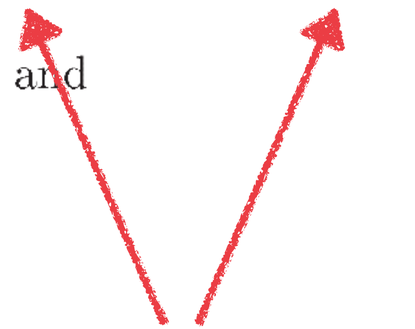

S in $D \vee \neg A_1 \vee \ldots \vee \neg A_n$

Abstract Redundancy

Γ is the set of inferences that makes up an inference system.

Abstract redundancy is defined:

```
locale redundancy_criterion = inference_system +
  fixes
    Rf :: "'a clause set ⇒ 'a clause set" and
    Ri :: "'a clause set ⇒ 'a inference set"
  assumes
    "Ri N ⊆ Γ" and
    "N ⊆ N' ⟹ Rf N ⊆ Rf N'" and
    "N ⊆ N' ⟹ Ri N ⊆ Ri N'" and
    "N' ⊆ Rf N ⟹ Rf N ⊆ Rf (N - N')" and
    "N' ⊆ Rf N ⟹ Ri N ⊆ Ri (N - N')" and
    "satisfiable (N - Rf N) ⟹ satisfiable N"
```

Standard Redundancy

```
definition Rf :: "'a clause set ⇒ 'a clause set" where
  "Rf N =
    {C. ∃DD. set_mset DD ⊆ N
         ∧ (∀I. I ⊨m DD ⟶ I ⊨ C)
         ∧ (∀D. D ∈# DD ⟶ D < C)}"
```

```
definition Ri :: "'a clause set ⇒ 'a inference set" where
  "Ri N =
    {γ ∈ Γ. ∃DD. set_mset DD ⊆ N
          ∧ (∀I. I ⊨m DD + side_prems_of γ ⟶ I ⊨ concl_of γ)
          ∧ (∀D. D ∈# DD ⟶ D < main_prem_of γ)}"
```

Theorem Proving Processes

```
definition "▷" :: "'a clause set ⇒ 'a clause set ⇒ bool" where
  "M ▷ N ⟷ N - M ⊆ concls_of (inferences_from M) ∧ M - N ⊆ Rf N"
```

The deduced clauses

The deleted clauses

Saturation up to redundancy:

```
definition saturated :: "'a clause set ⇒ bool" where
  "saturated N ⟷ inferences_from (N - Rf N) ⊆ Ri N"
```

Completeness of Ordered Ground Resolution

```
lemma saturated_complete_if:
  assumes
    "saturated N" and
    "¬ satisfiable N"
  shows "{#} ∈ N"
```

Ordered First-Order Resolution

Ordered resolution with selection

$$\frac{C_1 \vee A_{11} \vee \ldots \vee A_{1k_1} \quad \ldots \quad C_n \vee A_{1n} \vee \ldots \vee A_{nk_n} \quad \neg A_1 \vee \ldots \vee \neg A_n \vee D}{C_1\sigma \vee \ldots \vee C_n\sigma \vee D\sigma}$$

where σ is a most general simultaneous solution of all unification problems $A_{i1} = \ldots = A_{ik_i} = A_i$, where $1 \leq i \leq n$, and

(i) either $A_1, \ldots, A_n$ are selected in D, or else nothing is selected in D, $n = 1$, and $A_1\sigma$ is maximal in $D\sigma$,

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  ord_resolve:: "'a clause list ⇒ 'a clause ⇒ 's ⇒ 'a clause ⇒ bool"
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    ∀i < n. (CAs ! i) = (Cs ! i + (poss (AAs ! i))) ⟹
    ∀i < n. AAs ! i ≠ {#} ⟹
    Some σ = mgu (set_mset ` (set (map2 add_mset As AAs))) ⟹
    eligible σ As (D + negs (mset As)) ⟹
    ∀i. i < n ⟶ str_maximal_in (As ! i ·a σ) ((Cs ! i) · σ) ⟹
    ∀i < n. S (CAs ! i) = {#} ⟹
    ord_resolve CAs (D + negs (mset As)) σ (((∪# (mset Cs)) + D) · σ)"
```

Length of lists

Composition of clauses

Side conditions

Ordered First-Order Resolution

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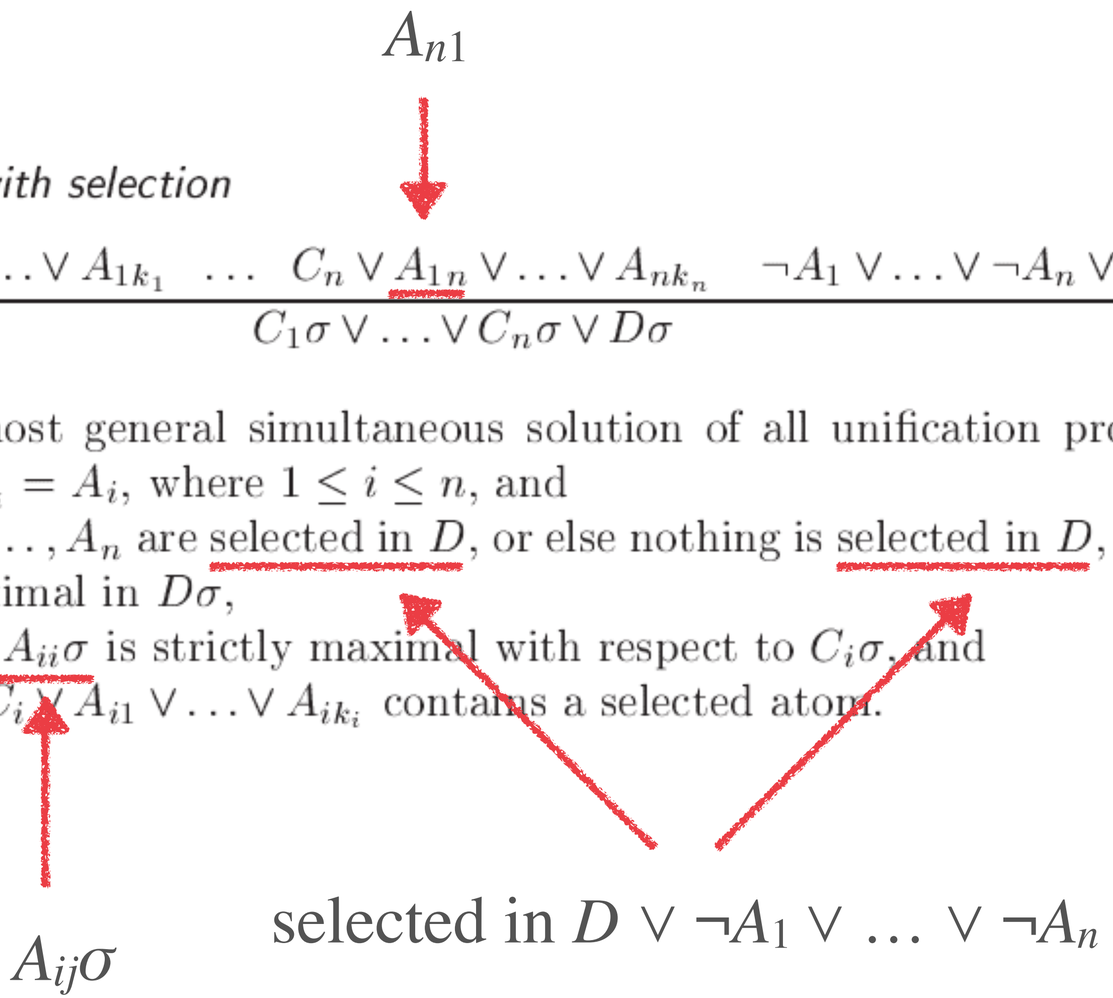

Prover

Tautology deletion

$$\mathcal{N} \cup \{C\} \mid \mathcal{P} \mid \mathcal{O} \implies \mathcal{N} \mid \mathcal{P} \mid \mathcal{O} \qquad \text{if } C \text{ is a tautology}$$

Forward subsumption

$$\mathcal{N} \cup \{C\} \mid \mathcal{P} \mid \mathcal{O} \implies \mathcal{N} \mid \mathcal{P} \mid \mathcal{O} \qquad \text{if some clause in } \mathcal{P} \cup \mathcal{O} \text{ subsumes } C$$

Backward subsumption

$$\mathcal{N} \mid \mathcal{P} \cup \{C\} \mid \mathcal{O} \implies \mathcal{N} \mid \mathcal{P} \mid \mathcal{O}$$
$$\mathcal{N} \mid \mathcal{P} \mid \mathcal{O} \cup \{C\} \implies \mathcal{N} \mid \mathcal{P} \mid \mathcal{O}$$
$$\text{if some clause in } \mathcal{N} \text{ properly subsumes } C$$

Forward reduction

$$\mathcal{N} \cup \{C \vee L\} \mid \mathcal{P} \mid \mathcal{O} \implies \mathcal{N} \cup \{C\} \mid \mathcal{P} \mid \mathcal{O}$$
$$\text{if there is a clause } D \vee L' \text{ in } \mathcal{P} \cup \mathcal{O} \text{ such that } \overline{L} = L'\sigma \text{ and } D\sigma \subseteq C$$

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Clause processing

$$\mathcal{N} \cup \{C\} \mid \mathcal{P} \mid \mathcal{O} \implies \mathcal{N} \mid \mathcal{P} \cup \{C\} \mid \mathcal{O}$$

Inference computation

$$\emptyset \mid \mathcal{P} \cup \{C\} \mid \mathcal{O} \implies \mathcal{N} \mid \mathcal{P} \mid \mathcal{O} \cup \{C\}$$

where $\mathcal{N}$ = all conclusions of inferences where one premise is C and the others are in $\mathcal{O}$

Prover

A state is a triple ($\mathcal{N}$,$\mathcal{P}$,$\mathcal{O}$) of sets of respectively

$\mathcal{N}$ew, $\mathcal{P}$rocessed, and $\mathcal{O}$ld clauses.

Let's look at three of the rules:

Forward reduction

$$\mathcal{N} \cup \{C \vee L\} \mid \mathcal{P} \mid \mathcal{O} \implies \mathcal{N} \cup \{C\} \mid \mathcal{P} \mid \mathcal{O}$$

if there is a clause $D \vee L'$ in $\mathcal{P} \cup \mathcal{O}$ such that $\overline{L} = L'\sigma$ and $D\sigma \subset C$

Clause processing

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A Simple Proof?

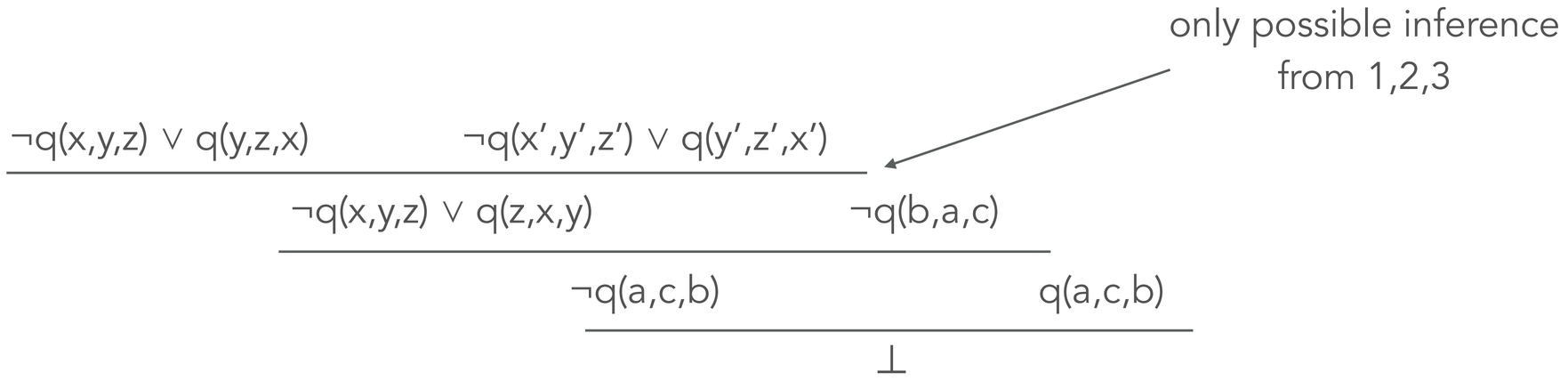
Consider the set containing and the selection function $S(C)=\varnothing$ for all C.

1) q(a,c,b)

2) ¬q(x,y,z) ∨ q(y,z,x)

3) ¬q(b,a,c)

with ordering q(c, b, a) > q(b, a, c) > q(a, c, b).



However, the prover will not do this inference!

Repairing the Incompleteness

Inference computation

$$\emptyset \mid \mathcal{P} \cup \{C\} \mid \mathcal{O} \implies \mathcal{N} \mid \mathcal{P} \mid \mathcal{O} \cup \{C\}$$

where $\mathcal{N}$ = all conclusions of inferences where one premise is C

and the others are in $\mathcal{O} \cup \{C\}$

Prover

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where $\mathcal{N}$ = all conclusions of inferences where one premise is C and the others are in $\mathcal{O} \cup \{C\}$

Repaired Prover, Formalized

```
inductive resolution_prover :: "'a state ⇒ 'a state ⇒ bool"
  tautology_deletion:
    "Neg A ∈# C ⟹ Pos A ∈# C ⟹ (N ∪ {C}, P, Q) ↝ (N, P, Q)"
| forward_subsumption:
    "(∃D ∈ P ∪ Q. subsumes D C) ⟹ (N ∪ {C}, P, Q) ↝ (N, P, Q)"
| backward_subsumption_P:
    "(∃D ∈ N. properly_subsumes D C) ⟹ (N, P ∪ {C}, Q) ↝ (N, P, Q)"
| backward_subsumption_Q:
    "(∃D ∈ N. properly_subsumes D C) ⟹ (N, P, Q ∪ {C}) ↝ (N, P, Q)"
| forward_reduction:
    "(∃D L'. D + {#L'#} ∈ P ∪ Q ∧ - L = L' ·l σ ∧ D · σ ≤# C) ⟹
      (N ∪ {C + {#L#}}, P, Q) ↝ (N ∪ {C}, P, Q)"
| backward_reduction_P:
    "(∃D L'. D + {#L'#} ∈ N ∧ - L = L' ·l σ ∧ D · σ ≤# C) ⟹
      (N, P ∪ {C + {#L#}}, Q) ↝ (N, P ∪ {C}, Q)"
| backward_reduction_Q:
    "(∃D L'. D + {#L'#} ∈ N ∧ - L = L' ·l σ ∧ D · σ ≤# C) ⟹
      (N, P, Q ∪ {C + {#L#}}) ↝ (N, P ∪ {C}, Q)"
| clause_processing:
    "(N ∪ {C}, P, Q) ↝ (N, P ∪ {C}, Q)"
| inference_computation:
    "N = concls_of (inferences_between Q C) ⟹
     ({}, P ∪ {C}, Q) ↝ (N, P, Q ∪ {C})"
```

Completeness

Bachmair and Ganzinger prove completeness under the assumption of fairness.

A sequence of states is fair if

- no clause eventually always stays in $\mathscr{N}$, and
- no clause eventually always stays in $\mathscr{P}$.

The big picture of the proof is roughly:

1. Assume the initial set of clauses is unsatisfied.
2. Project the sequence of states down to the ground level.
3. By fairness, any non-redundant clause eventually gets old.
4. Therefore the grounding of the old clauses is saturated.
5. Using completeness of ground resolution we get the empty clause.

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Any Nonredundant Ground Clause Eventually Gets Old?

A flawed proof of this lemma is provided by the authors.

The big picture of their proof is:

1. Consider a non-redundant ground clause C in the grounding of $\mathcal{N}$ or $\mathcal{P}$.
2. It is the instance of some clause D in $\mathcal{N}$ or $\mathcal{P}$.
3. By fairness D cannot stay forever in $\mathcal{N}$ or $\mathcal{P}$ and thus must have been removed by some rule.
4. By inspection of the rules we can see that D must eventually be in $\mathcal{O}$.
5. Therefore C must be in the grounding of $\mathcal{O}$.

Counterexample to 4:

({p(x)}, {p(f(y))}, {}) $\Longrightarrow$ ({p(x)}, {}, {})

Backward subsumption

$$\mathcal{N} \mid \mathcal{P} \cup \{D\} \mid \mathcal{O} \implies \mathcal{N} \mid \mathcal{P} \mid \mathcal{O}$$

if some clause in $\mathcal{N}$ properly subsumes D

Any Nonredundant Ground Clause Eventually Gets Old?

smallest w.r.t. subsumption

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5. Therefore C must be in the grounding of $\mathcal{O}$.

We know D exists since strict subsumption is well founded.

Completeness of FO Ordered Resolution Prover

```
theorem completeness:
  assumes
    renaming: "∧ρ C. is_renaming ρ ⟹ Sel (C · ρ) = Sel C · ρ"
  assumes
    "derivation (op ⇝) S" and
    "fair_state_seq S" and
    "¬ satisfiable (grounding_of_state S₀)" and
  shows "{#} ∈ clss_of_state S∞"
```

What Was Tricky?

Many lemmas and their proofs have flaws:

- Lemma 3.4 contains a wrong claim.
- Lemma 3.7 has a counter example, but the lemma is fortunately never used.
- Section 4.1 contains results that require soundness, but it is claimed that consistency-preservability is enough.
- Theorem 4.3's proof does not cover all cases.
- Lemma 4.10 uses a selection function, but it is not clear which.
- Lemma 4.10 concerns the extension of a proof system, but it is not clear which.
- Lemma 4.10 is only proved for finite sequences, but later used for infinite ones.
- Lemma 4.12 has a 2 sentence proof. In Isabelle the proof is about 500 LOC.

General issue: Lemmas are stated, but not referenced later.

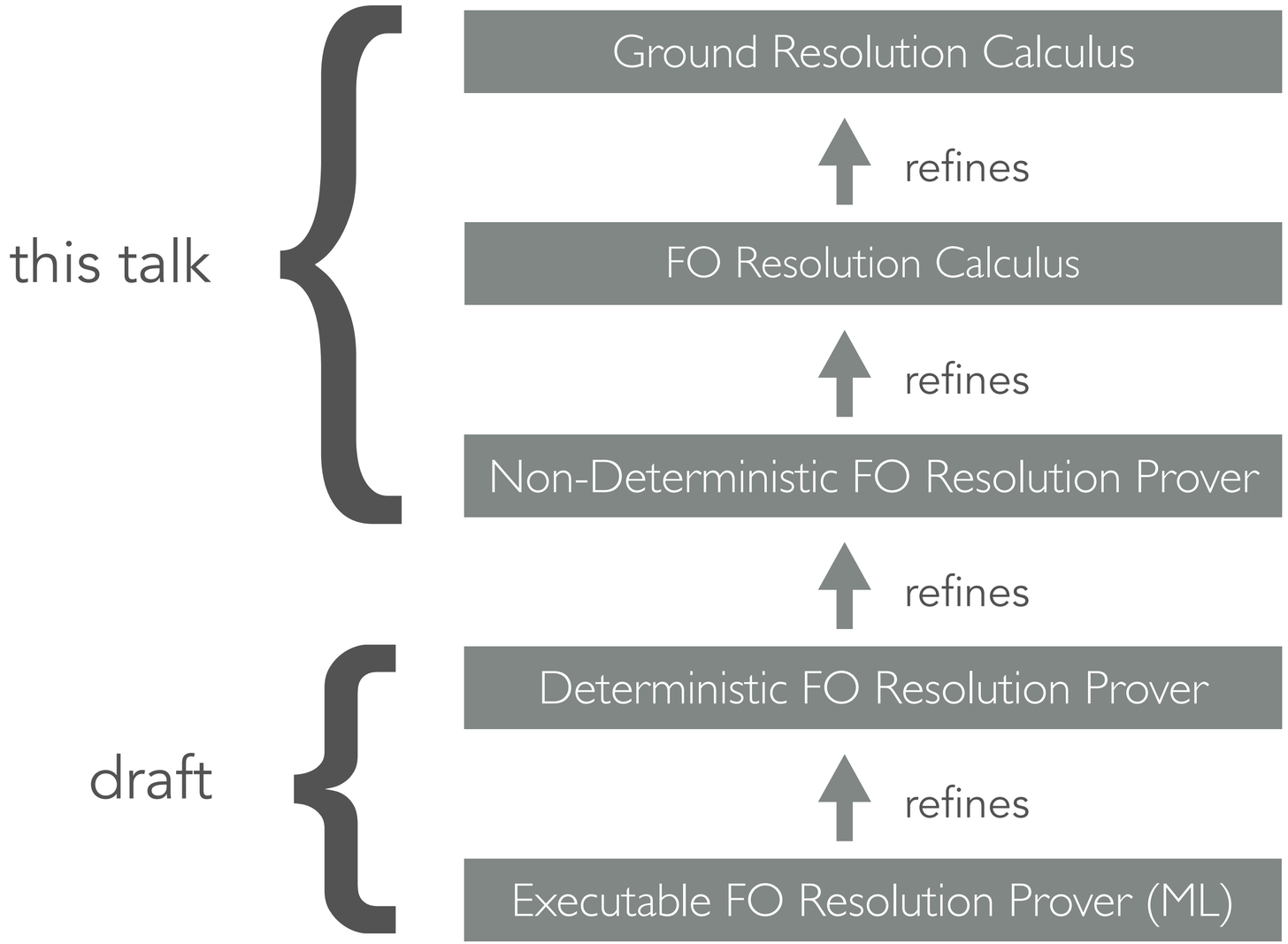
What Was Tricky?

Procedure for dealing with these problems:

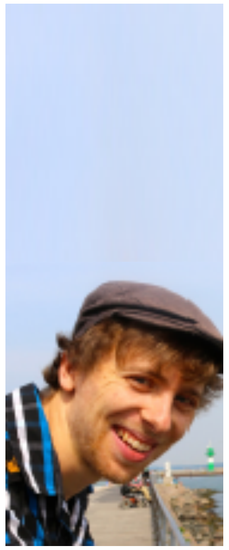
1. Rewrite the chapter's proofs to handwritten pseudo-Isabelle.
2. Fill in the gaps and write where lemmas are used.
3. Turn the pseudo-Isabelle into real Isabelle, but with **sorry** instead of proofs.
4. Replace the **sorry**s with proofs.

Backtrack when necessary.

Refinement Chain



What Can We Learn from Formalization?



I have a question about Chapter 2 of the Handbook of AR. On p. 45, they introduce a function S_M to select literals in ground instances of clauses in M. The definition says "(ii) S_M(C) = S(C), if C is not in K."
Do you know why they define the function for clauses not in K? Is it because they don't want to leave S_M underspecified or does this serve a special purpose?

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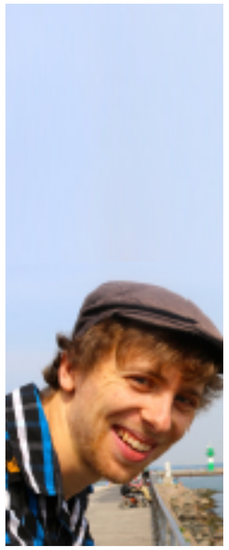

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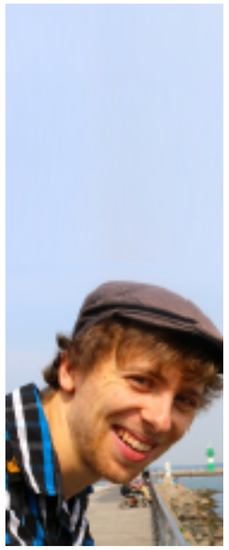

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I tried to change the definition in the formalization to return {} if C is not in K. With this definition the key properties also hold, and **the proof goes through**.

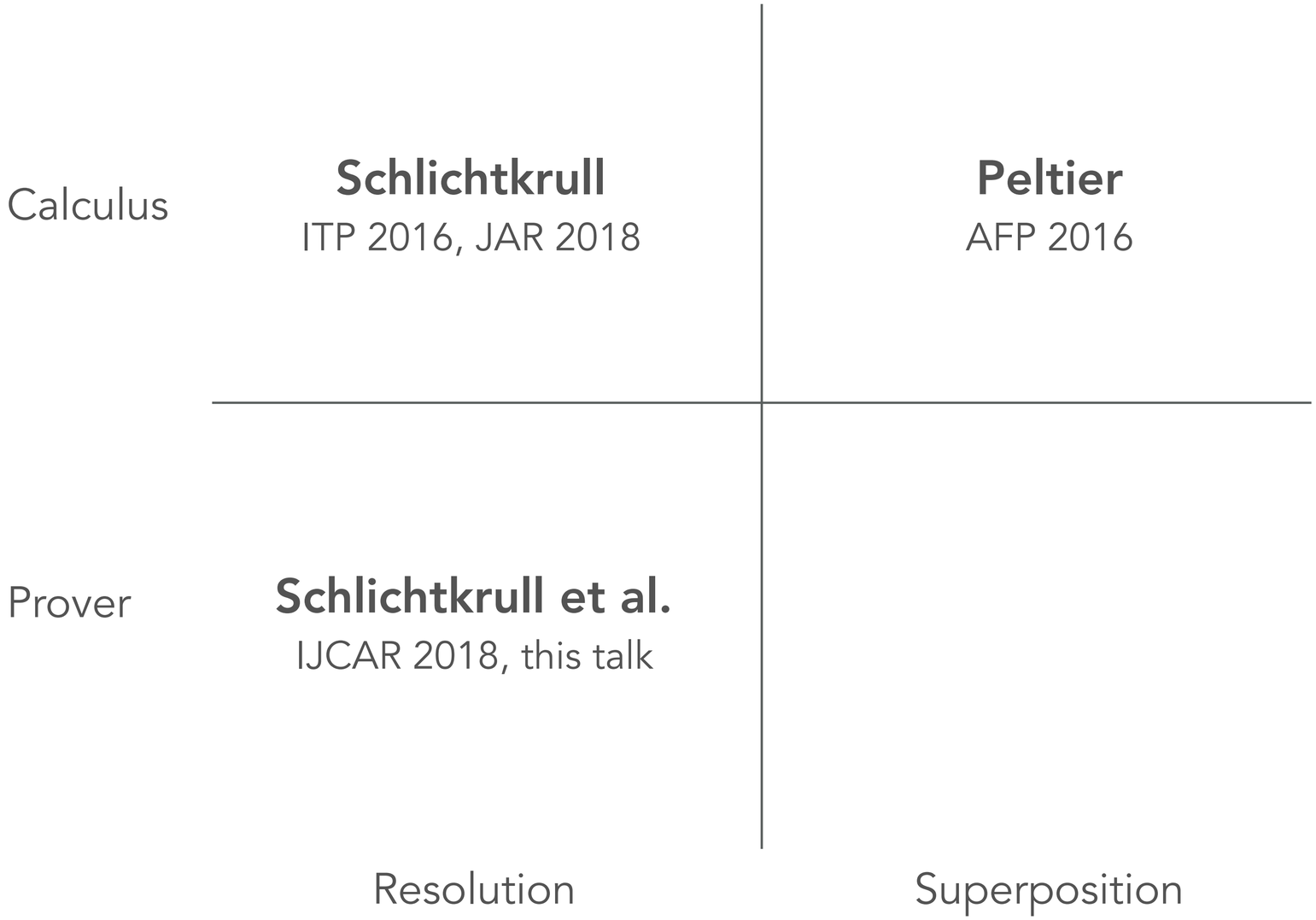
Suitability of Isabelle

Isabelle was entirely suitable for this development.

We benefitted from

- codatatypes
- Isabelle/jEdit
- Isar proofs
- locales
- Sledgehammer.

Related Work



Conclusion

We have formalized Bachmair and Ganzinger's prover.

The chapter withstood the test of formalization after

- allowing the prover to do self-inferences
- repairing some of the proofs.

The formalization

- clarifies the chapter's content
- contributes to the effort of making useful tools for developing AR metatheory.