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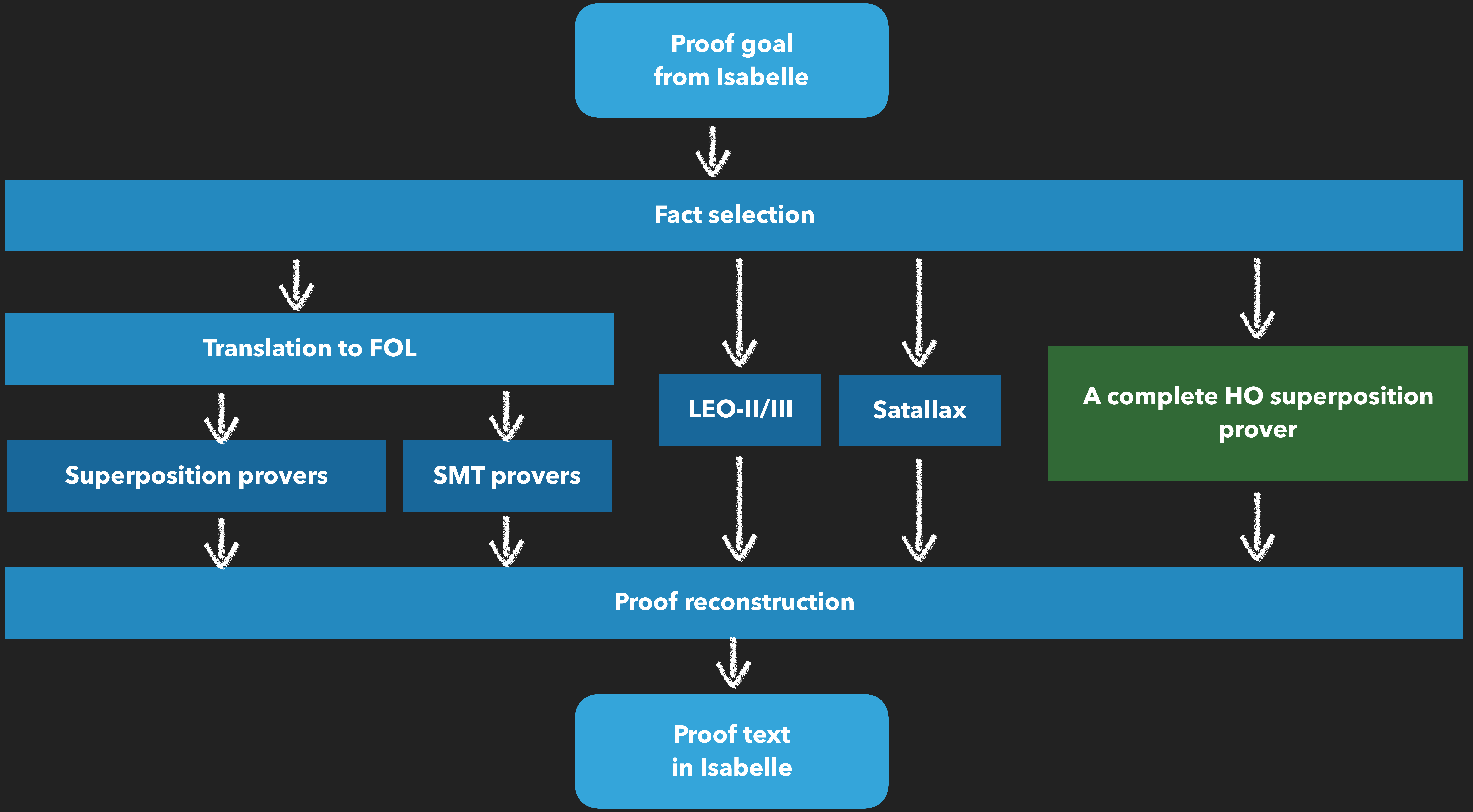
SUPERPOSITION FOR LAMBDA-FREE HIGHER-ORDER LOGIC

Motivation: Sledgehammer



DESIGN PRINCIPLE: BE GRACEFUL

HO superposition on first-order problems should coincide with FO superposition

Our way to higher-order superposition

HOL

predicate-free HOL

λ-free HOL / applicative FOL

FOL

partial application & applied variables

λ-expressions / comprehension axioms

boolean formulas nested in terms

Translation to FOL: applicative encoding

$f\ (H\ f)$ is translated to $\mathrm{app}(f, \mathrm{app}(H, f))$

λ-free HOL — FOL

NOT GRACEFUL!

Term orders for λ-free HOL

Compatibility with arguments?

$t > s \Rightarrow t\,u > s\,u$

KBO without argument coefficients

LPO

KBO with argument coefficients

Yes:
Petar's talk

No:
This talk

The superposition rule

$$\frac{D \vee t = t' \qquad\qquad C \vee (\neg)\, s[u] = s'}{(D \vee C \vee (\neg)\, s[t'] = s')\sigma} \quad \sigma = \mathrm{mgu}(t,u)$$

+ order conditions

Superposition only at argument subterms

Argument subterms: $\underline{f\ \underline{a}\ (\underline{h\ \underline{b}\ \underline{c}})}$

Prefix subterms: $\underline{\underline{f}\ a}\ (\underline{\underline{h}\ b}\ c)$

$$\frac{g = f \qquad\qquad g\ a \neq b}{f\ a \neq b}\ \cancel{\textsc{Sup}}$$

Argument congruence rule

$$\frac{C \lor t = s}{C \lor t\,X = s\,X}\ \textsc{ArgCong}$$

Example:

$$\textsc{Sup}\ \frac{\textsc{ArgCong}\ \dfrac{g = f}{g\,X = f\,X} \qquad g\,a \neq b}{f\,a \neq b}$$

Argument congruence rule

$$\frac{C \vee t = s}{C \vee t\,X = s\,X}\ \textsc{ArgCong}$$

BUT ISN'T THIS RULE ALWAYS REDUNDANT?

Floor encoding

Encode **ground λ-free HOL** terms into **FOL:**

$$\lfloor \mathrm{f} \rfloor = \mathrm{f}_0$$
$$\lfloor \mathrm{f\ a} \rfloor = \mathrm{f}_1(\mathrm{a}_0)$$

Redundancy is defined with respect to this encoding.

Floor encoding

Example:

$$\textsc{ArgCong}\ \frac{g = f}{g\ X = f\ X}$$

$$\frac{g_0 = f_0}{g_1\ a_0 = f_1\ a_0}$$

Not redundant!

What changes in the proof?

Refutational completeness:

Let N be saturated up to redundancy, $\bot \notin N$.

Then N has a model.

Proof sketch for FOL:

$$
\begin{tikzcd}
N \arrow[d] & & \text{model of } N \\
G(N) \arrow[rr, "\text{model construction}"'] & & \text{model of } G(N) \arrow[u]
\end{tikzcd}
$$

What changes in the proof?

Refutational completeness:

Let N be saturated up to redundancy, $\bot \notin N$.

Then N has a model.

Proof sketch for λ-free HOL:

$$\begin{tikzcd}
N \arrow[d] & & \text{model of } N \\
G(N) \arrow[d] & & \text{model of } G(N) \arrow[u] \\
\lfloor G(N) \rfloor \arrow[rr, "\text{model construction}"'] & & \text{model of } \lfloor G(N) \rfloor \arrow[u]
\end{tikzcd}$$

Issue: superposition at variables

Example:

$$C = \dots X \dots X\, a \dots$$

Given g > f, it is unclear whether X := g or X:= f will yield the smaller clause

Solution #1:
purifying calculus

$$\dots X\, \overline{\mathbf{u}} \dots X\, \overline{\mathbf{v}} \dots$$

is purified to

$$\dots X\, \overline{\mathbf{u}} \dots Y\, \overline{\mathbf{v}} \dots \vee X \neq Y$$

if $\overline{\mathbf{u}} \neq \overline{\mathbf{v}}$

Solution #2:
nonpurifying calculus

Perform superpositions at variables if the order situation is unclear

Evaluation of our prototype

using the Zipperposition theorem prover

# unsat	TPTP benchmarks FO	TPTP benchmarks HO	Judgment Day λ-free HOL benchmarks 32 facts	Judgment Day λ-free HOL benchmarks 512 facts
first-order mode	**181**	-	-	-
applicative encoding	151	**677**	**873**	843
purifying calculus	180	647	851	**908**
nonpurifying calculus	179	669	866	889

In summary

- We developed refutationally complete calculi for λ-free HOL
- They reduce the gap between HO proof assistants and superposition provers
- They are promising as a stepping stone towards a HO superposition calculus